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ON THE RATE OF CONVERGENCE TO NORMALITY OF ESTIMATES OF REGRESSION COEFFICIENT FOR ASSOCIATED RANDOM FIELDS

The rate of convergence in the central limit theorem is studied for the least square estimates of regression coefficients of associated random fields. The upper optimal estimate is obtained.

Key words: least square estimates, central limit theorem, associated random fields

Introduction. The rate of convergence in the central limit theorem for fields of weakly dependent random variables has been studied in many papers (see, for instance, Tikhomirov [8], Synklodas [7]). Conditions of $m-$ dependence and strong mixing are practically difficult to verify, i.e., to justify in practical situations.

One of the more practically applicable approach to modeling weakly dependent observations has been introduced by Esary, Proschan and Walkup [4] described by the concept of association or positive dependence. Recall that a finite collection $\xi=\left(\xi_1,\xi_2,...,\xi_m\right)$ of real-valued random variables is called associated if, for any coordinate-wise non decreasing functions $f,g:R^m\to R$ such that $Ef(\xi)g(\xi)$, $Ef(\xi)$ and $Eg(\xi)$ are finite, one has

$$\operatorname{cov}\left(f(\xi),g(\xi)\right)\geq 0.$$

For an infinite number of real-valued random variables to be associated means that all finite subfamilies have this property. A family of independent random variables is associated [4]. Moreover, if $\eta=\left(\eta_1,...,\eta_m\right)$ is a collection (in particular a singleton) of associated variables and $h_k:R^m\to R,\ k=1,2,...,n$ are coordinate-wise non-decreasing functions, then $h_1(\eta),...,h_n(\eta)$ are associated (see the same).

The central limit theorem for strictly stationary fields of associated random variables on Z^d has been proved by Newman [6]. Rates of convergence in the central limit theorem for associated random variables have been obtained by Birkel [1]. Rates of convergence in the central limit theorem for random fields have been obtained by Bulinskii [2].

1. Preliminary. Consider a random fields $\xi_j=\theta\varphi_j+\varepsilon_j$, where $\varphi_j:Z^d\to R^1$ is a known function, $\varepsilon_j:Z^d\to R^1$ is a random field with mean 0. Let a system of bounded increasing sets A_n be singled out in Z^d and let θ be an unknown parameter to be estimated from observations of the random fields ε_j on the sets A_n.

Let $\theta\ =\sum\limits_{j\in A_n}\dfrac{\xi_j\varphi_j}{d_n^2}$ be the least square estimates for the parameter θ,

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where $d_n^2 = \sum_{j \in A_n} \varphi_j^2$. Obviously, θ_n is an unbiased estimate for θ. The theorem that establishes the asymptotic normality of least square estimate for regression coefficient of associated random fields was proved by Koval' [5].

2. Main result. In this paper, the rate convergence in the central limit theorem is studies for such estimates.

Theorem. Let a field $\{\varepsilon_j : j \in Z^d\}$ of associated random variables with zero mean satisfy the following assumptions:

1) $\sup_j E|\varepsilon_j|^s < \infty$ for some $s \in (2,3]$;

2) $u(n) = o(e^{-\lambda n})$ as $n \to \infty$ for some $\lambda > 0$;

3) $\varphi_j \geq 0, \quad \sup_j \frac{\varphi_j}{d_n} \leq \frac{L}{|A_n|^{\frac{1}{2}}}$;

4) $\lim_{n\to\infty} \frac{\sigma_n^2}{d_n^2} = \sigma_0^2$.

Then

$$\Delta_n = \sup_x \left| P\left(d_n\left(\theta_n^* - \theta\right) < x\right) - \varphi(x) \right| \leq$$

$$\leq B\left(\left(\ln|A_n|\right)^{d(s-1)} |A_n|^{-\frac{s-2}{2}} + \left| 1 - \frac{\sigma_n^2}{d_n^2 \sigma_0^2} \right| \right),$$

where B being independent of $|A_n|$ positive constant, $u(n) = \sup_{j \in Z^d} \sum_{q \in Z^d : |q-j| \geq n} \operatorname{cov}(\varepsilon_j, \varepsilon_q)$, $|y| = \max_{1\leq k\leq d} |y_k|$ for $y \in R^d$, $|A_n|$ is the number of points in A_n, $\sigma_n^2 = E\left(\sum_{j \in A_n} \varepsilon_j \varphi_j \right)^2$, $\varphi(x)$ – denotes distribution function of normal law with $N(0, \sigma_0^2)$.

The theorem is proved by using the method of characteristic functions .

Let us introduce the notation:

$$V_{j,n}^l = A_{j,n}^{l-1}, \quad x_j = \varepsilon_j \varphi_j / (d_n \varphi_0), \quad S_{j,n}^0 = S_n = \sum_{j \in A_n} x_j, \quad S_{j,n}^{(l)} = \sum_{k \in A_{j,n}^l} x_k,$$

$$A_{j,n}^{(0)} = A_n, \quad A_{j,n}^l = \{q \in A_n : |q-j| > lm\}, \quad U_{j,n}^l = \sum_{k \in V_{j,n}^l} x_k,$$

$$\xi_j = \exp\{itU_{j,n}^l\} - 1, \quad \zeta_j^{(r)} = x_j \prod_{l=1}^r \xi_j^l, \quad f_n(t) = E\exp(itS_n).$$

Then $f_n'(t) = i \sum_{j \in A_n} Ex_j \exp\{itS_{j,n}^{(0)}\}$. Using the result from [8], we arrive a the representation

$$f_n' = i\sum_{j\in A_n} Ex_j \exp\left\{itS_{j,n}^r\right\} + i\sum_{r=1}^{k}\sum_{j\in A_n} E\zeta_j^r\left(\exp\left\{itS_{j,n}^{r+1}\right\} - f_n(t)\right) +$$

$$+ if_n(t)\sum_{r=1}^{k}\sum_{j\in A_n} E\zeta_j^n + i\sum_{j\in A_n} E\zeta_j^{(k+1)}\exp\left\{itS_{j,n}^{(k+1)}\right\}. \quad (1)$$

Denoting

$$c_s = c_s(A_n) = \max_{j\in A_n}\left(E\left|\varepsilon_j\right|^s\right)^{\frac{1}{s}},$$

$$b_s(m,r) = b_s(A_n,m,r) = \max_{j\in A_n}\max_{1\le l\le r}\left(E\left|U_{j,n}^{(l)}\right|^s\right)^{\frac{1}{s}}_{j\in A_n}.$$

Denote $\theta_i(t)$ – a complex-valued function, $\left|\theta_i(t)\right| \le 1$. The proof of the theorem based on lemmas.

Lemma 1. For all $t\in R$

$$i\sum_{j\in A_n} Ex_j \exp\left\{itS_{j,n}^{(1)}\right\} \le 2A\theta_1(t)|t|u(m).$$

Now, let for $m, r\in N$

$$Q(m,1) = Q(m,2) = 0,\ Q(m,r) = \max_{j\in A_n}\max_{1\le l\le r-2}\sum_{p=l+2}^{r}\operatorname{cov}\left(U_{j,n}^{(l)}, U_{j,n}^{s}\right),\ r\ge 3.$$

Lemma 2. Let $t\in R$ be such, that $\dfrac{|t|Lb_s(m,k+1)}{|A_n|^{\frac{1}{2}\sigma_0}} < \dfrac{1}{4}$, than

$$E\left|\zeta_j^1\right| \le (3m)^{\frac{d}{2}}\frac{L^2|t|c_s}{|A_n|\sigma_0^2}(u(0))^{\frac{1}{2}}.$$

Lemma 3.

$$T_1(m,r) := E\left|\zeta_j^r\right| \le \frac{Lc_s}{\left|A^n\right|^{\frac{1}{2}}}\left(\left(\prod_{l=1}^{r}\left(E\left|\xi_j^{(l)}\right|^s\right)^{\frac{1}{s}} + r^{\frac{2}{s}}2^r\left(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0}\right)^{\frac{4}{s}}Q(m,r)^{\frac{2}{s}} + \right.\right.$$

$$+8r^{\frac{1}{s}}2^{\frac{r}{2}}c_s\left(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0}\right)^{\frac{2}{s}}Q(m,r)^{\frac{1}{s}}) + 8u(m)^{\frac{s-2}{s-1}}\left(\frac{Lc_s}{\left|A^n\right|^{\frac{1}{2}}\sigma_0}\right)^{s-1} +$$

$$\left.+|t|^{\frac{s-1}{s}}\left(\frac{L}{|A_n|^{\frac{1}{2}}\sigma_0}\right)^{\frac{3(s-1)}{s}}\right)\left(r^{\frac{1}{s}}2^r\left(\frac{|t|L}{|A_n|^{\frac{1}{2}}}\right)^{\frac{2}{s}}Q(m,r)^{\frac{1}{s}} + 2^{-\frac{r}{2}}\right)\Bigg).$$

Lemma 4. For all $t\in R$

$$i\sum_{j\in A_n} E\zeta_j^{(1)} = -t + t\left|1 - \frac{\sigma_n^2}{d_n^2\sigma_0^2}\right| + t\theta_2(t)c_1u(m) +$$

$$+\ 2\theta_3(t)c_s\left(\frac{|t|Lb_s(m,1)}{|A_n|^{\frac{1}{2}}\sigma_0}\right)^{s-1}\frac{c_2}{|A_n|^{\frac{1}{2}}},$$

where c_i – are bounded variables.

Lemma 5. For all $r\in N$, $j\in A_n$ and t satisfying $\frac{|t|Lb_s(m,k+1)}{|A_n|^{\frac{1}{2}\sigma_0}}<\frac{1}{4}$, the following inequalities hold

$$\left|\operatorname{cov}\left(\xi_j^n,\exp\left\{itS_{j,n}^{(r+1)}\right\}\right)\right|\le\frac{L}{|A_n|^{\frac{1}{2}}\sigma_0}2^{2r+2}\left(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0}u(m)+\right.$$

$$\left.+\delta(m.r)^{\frac{s-1}{s}}\left(\left(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0}\right)^2+c_c^s\left(\frac{L}{|A_n|^{\frac{1}{2}}}\right)^s\right)\right).$$

Lemma 6. For all $m,r\in N$, the following estimates are valid:

$$b_s(m,r)\le d(3m)^d\,r^{d-1}c_2\frac{L}{|A_n|^{\frac{1}{2}}\sigma_0},$$

$$\delta(m,r)\le d(3rm)^d\frac{L}{|A_n|^{\frac{1}{2}}\sigma_0}\sum_{n\ge m}u(n),$$

$$Q(m,r)\le d(3rm)^d\frac{L^2}{|A_n|^{\frac{1}{2}}\sigma_0}\sum_{n\ge m}u(n).$$

Lemmas 1–6 are proved by using Holder inequality, properties associating random variables, Berry–Esseen's inequality, Birkel's lemma [1, 3].

Based on lemmas 1–6 equation (1), may be rewritten in following form

$$f_n'(t)=-tf_n(t)+t\left|1-\frac{\sigma_n^2}{d_n^2\sigma_0^2}\right|f_n(t)+\theta_4(t)A(t)f_n(t)+\theta_s(t)B(t),\qquad(2)$$

where

$$A(t)=|A_n|^{-\frac{1}{2}}t^2\left(\ln|A_n|\right)^{2d}+|t|^{s-1}|A_n|^{-\left(\frac{s}{2}-1\right)}\left(\left(\ln|A_n|\right)^{d(s-1)}\right),$$

$$B(t)=|A_n|^{-\frac{1}{2}}+|A_n|^{-\frac{1}{2}}t^2\left(\ln|A_n|\right)^{2d}.$$

By integrating (2) for $|t|\le T_0$ and applying Berry–Esseen's inequality one, can easily deduce (see [8]), that

$$\Delta_n\le c\left\{|A_n|^{-\frac{1}{2}}\left(\ln|A_n|\right)^{2d}+|A_n|^{-\left(\frac{s}{2}-1\right)}\left(\ln|A_n|\right)^{d(s-1)}+\right.$$

$$\left.+\left|1-\frac{\sigma_n^2}{d_n^2\sigma_0^2}\right||A_n|^{-\frac{1}{2}}\ln|A_n|+|A_n|^{-\frac{1}{2}}\left(\ln|A_n|\right)^{2d}\right\}.$$

Then the estimate (1) is true, and the theorem is proved. Obtained estimate is the upper, optimal in some sense Berry–Essen's type estimate, the rate of convergence in the central limit theorem.

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ШВИДКІСТЬ ЗБІЖНОСТІ ДО НОРМАЛЬНОГО ЗАКОНУ ОЦІНКИ КОЕФІЦІЄНТА РЕГРЕСІЇ АСОЦІЙОВАНИХ ВИПАДКОВИХ ПОЛІВ

Досліджено швидкість збіжності в центральній граничній теоремі для оцінювання найменших квадратів коефіцієнта регресії асоційованих випадкових полів. Отримано верхню оптимальну оцінку швидкості збіжності.

Ключові слова: *оцінка найменших квадратів, центральна гранична теорема, асоційовані випадкові поля*

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