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ON PROPERTIES OF POSETS OF *MM*-TYPE (1, 3, 4)

We calculate the coefficients of transitiveness for all posets of MM-type (1, 3, 4) (*i.e. posets, which are minimax equivalent to the poset* (1, 3, 4)).

Key words: *neighboring elements, coefficient of transitiveness, minimax equivalence, anti-isomorphism, MM-type, height of a poset, nodal element, dense subset.*

Introduction. In 1972 P. Gabriel [16] introduced the notion of representation of a finite quiver $Q=(Q_0,Q_1)$ (Q_0 and Q_1 denote the sets of vertices and arrows, respectively) and an integral quadratic form $q_Q:\ \mathbb{Z}^n\to\mathbb{Z},\ n=|Q_0|$, called by him the quadratic Tits form of the quiver Q:

$$q_Q(z)=q_Q(z_1,...,z_n):=\sum_{i\in Q_0} z_i^2-\sum_{i\to j} z_i z_j\ ,$$

where $i\to j$ runs through the set Q_1. He proved that the quiver Q has finite representation type over a field k if and only if its Tits form is positive (i.e., takes positive value on any nonzero vector).

The above quadratic form is naturally generalized to a finite poset $0\notin S$:

$$q_S(z)=z_0^2+\sum_{i\in S} z_i^2+\sum_{i<j,\,i,j\in S} z_i z_j - z_0\sum_{i\in S} z_i\ .$$

In [15] Yu. A. Drozd showed that a poset S has finite representation type if and only if its Tits form is weakly positive, (i.e., takes positive value on any nonzero vector with nonnegative coordinates); representations of posets were introduced by L. A. Nazarova and A. V. Roiter [17].

Note that, in contrast to quivers, the sets of posets with weakly positive and with positive Tits form do not coincide. Posets with positive Tits forms were studied by the authors in many papers (see e.g. [1–10]).

In particular, in [6] it is introduced the notion of P-critical poset (a poset S is said to be P-critical if its Tits form is not positive, but that of any proper subset of S is positive). Such posets are analogs of the extended Dynkin diagrams. They are classified in [6].

Combinatorial properties of P-critical posets were studied by the authors in [11]. Analogous properties were studied in [12–14] for the posets of MM-type (2, 2, 3), (1, 3, 5) and (1, 2, 6).

In this paper we continue these investigations.

1. Coefficients of transitiveness. In this section we recall the notion of coefficient of transitiveness of a poset [11].

Let S be a finite poset and $S^2_< := \{(x,y)\mid x,y\in S, x<y\}$. If $(x,y)\in S^2_<$ (x, y) and there is no z satisfying $x<y<z$, then x and y are called *neighboring*. We put $n_w=n_w(S):=|S^2_<|$ and denote by $n_e=n_e(S)$ the number of pairs of neighboring elements. On the language of the Hasse diagram $H(S)$ (that represents S in the plane), n_e is equal to the number of all its edges and n_w to the number of all its paths, up to parallelity, going bottom-up (two path is

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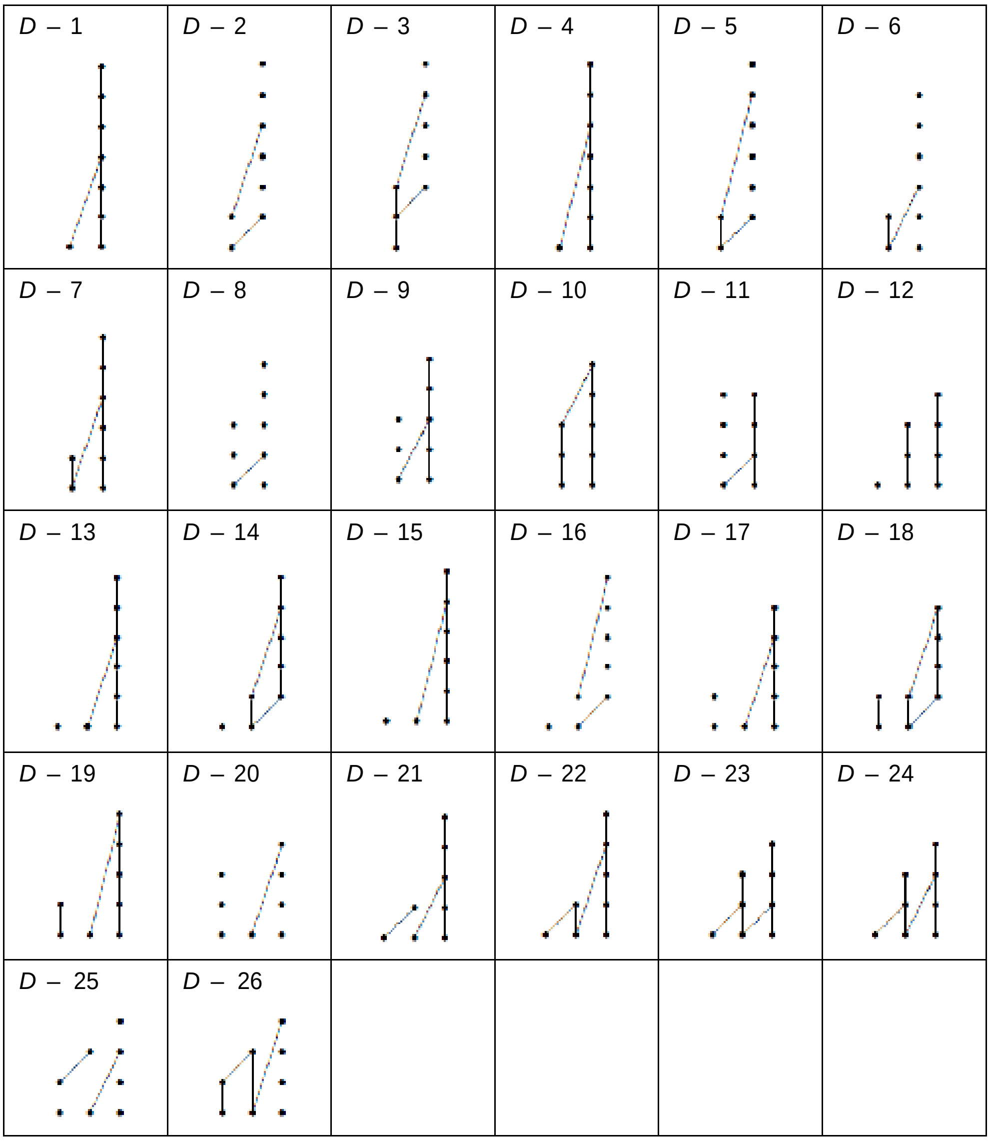
called parallel if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w is called *the coefficient of transitiveness of* S. In the case $n_w = 0$ (then $n_e = 0$) $k_t = 0$.

2. The posets of MM-type (1, 3, 4). For a fixed poset P we say that a poset S is of the MM-type P if S is (min, max)-equivalent to P (the notion of (min, max)-equivalence was introduced in [1]; see also [6]). Since some time we have been used the term *minimax equivalence*.

The poset (1, 3, 4) (the disjoint union of chains of length 1, 3 and 4) is an element with trivial group of automorphisms in the set of 1-oversupercritical posets (see [2]). The posets of MM-type (1, 3, 4) were classified in [2]. They are given (up to isomorphism and anti-isomorphism) by the following table.



3. Main result. In [11] the authors calculate the coefficient of transitiveness for all *P*-critical posets. Analogous result was obtained in [13] and [14] for the posets of *MM*-type (2, 2, 3) and (1, 2, 6). In this paper we do it for the posets of *MM*-type (1, 3, 4).

We write all the coefficients of transitiveness k_t up to the second decimal place.

Theorem. *The following holds for posets* (*D*-1)–(*D*-26) *of the MM-type* (1, 3, 4):

N	n_e	n_w	k_t	N	n_e	n_w	k_t	N	n_e	n_w	k_t
1	7	25	0,72	10	7	16	0,56	19	6	12	0,5
2	8	25	0,68	11	7	15	0,53	20	6	10	0,4
3	8	25	0,68	12	5	9	0,44	21	7	15	0,53
4	7	24	0,71	13	6	18	0,67	22	7	14	0,5
5	8	24	0,67	14	7	18	0,61	23	7	14	0,5
6	7	20	0,65	15	6	17	0,65	24	7	13	0,46
7	7	19	0,63	16	7	17	0,59	25	7	12	0,42
8	7	17	0,59	17	6	13	0,54	26	7	11	0,36
9	7	16	0,56	18	7	13	0,46				

The proof is carried out by direct calculations.

Recall that *height of a poset* S is, by definition, the greatest order of a linear ordered subposet of *S*. An element of *S*, which is comparable with all the others elements, is called *nodal*. A subposet T of S is called *dense* if from $x < y < z$, where $x, z \in T$, it follows that $y \in T$.

Corollary 1. Let S be a poset of the form *Di such that* $k_t(S) > k_t(T)$ *for all other such posets* T. *Then* $h(S) \geq h(T)$.

Corollary 2. *A poset of MM-type (1, 3, 4) has the largest coefficient of transitiveness if and only if it contains a dense subposet with four nodal element.*

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ПРО ВЛАСТИВОСТІ ЧАСТКОВО ВПОРЯДКОВАНИХ МНОЖИН *ММ*-ТИПУ (1, 3, 4)

Обчислено коефіцієнти транзитивності для всіх частково впорядкованих множин ММ-типу (*1, 3, 4*) (*тобто частково впорядкованих множин, які мінімаксно еквівалентні частково впорядкованій множині* (*1, 3, 4*)).

Ключові слова: *сусідні елементи, коефіцієнт транзитивності, мінімаксна еквівалентність, антиізоморфізм, ММ-тип, висота ч.в. множини, вузловий елемент, щільна підмножина.*

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