EXISTENCE RESULTS FOR THE DARBOUX PROBLEM FOR HYPERBOLIC DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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In this paper we investigate the existence of solutions to the Darboux problem for a third order hyperbolic differential and functional differential inclusion with nonconvex-valued right-hand side. We shall rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler and on Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued operators with nonempty closed and decomposable values.

1 Introduction

This paper deals with the existence of solutions to the Darboux problem for third order hyperbolic differential and functional differential inclusions in Banach spaces. In Section 3 we consider the Darboux problem for the hyperbolic differential inclusion:

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} \in F(x,y,z,u), (x,y,z) \in \mathcal{D} = J_a \times J_b \times J_c = [0,a]\times[0,b]\times[0,c] \quad (1)$$

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$$\begin{cases} u(x,y,0) = f(x,y), & (x,y) \in D_1 = [0,a] \times [0,b], \\ u(0,y,z) = g(y,z), & (y,z) \in D_2 = [0,b] \times [0,c], \\ u(x,0,z) = h(x,z), & (x,z) \in D_3 = [0,a] \times [0,c], \end{cases} \tag{2}$$

where $F : \mathcal{D} \times E \longrightarrow \mathcal{P}(E)$ is a given multivalued map, $f : D_1 \to E,\ g : D_2 \to E,\ h : D_3 \to E,\ a > 0,\ b > 0, c > 0$ and $(E, |\cdot|)$ a real separable Banach space, which satisfy the conditions

$$\begin{cases} f(x,0) = h(x,0) = v^1(x), & x \in [0,a], \\ f(0,y) = g(y,0) = v^2(y), & y \in [0,b], \\ g(0,z) = h(0,z) = v^3(z), & z \in [0,c], \\ v^1(0) = v^2(0) = v^3(0) = v^0. \end{cases}$$

This study was motivated by several papers which deal with the Darboux problem for third order hyperbolic equations [4, 5, 8, 9, 10, 11, 12, 19, 20, 21, 25, 26]. Other results on the Darboux problem for hyperbolic differential equations can be found in the book by Kamont [17] and the references therein. Very recently the problem (1)-(2) was studied by Teodoru ([24, 27, 28]) in the case of a convex multivalued right hand side. In this paper, we drop the convex condition and we shall give existence results for the problem (1)-(2) with a nonconvex-valued right-hand side. We shall present two results. In the first one we rely on a fixed point theorem for contraction multivalued maps, due to Covitz and Nadler [6] and for the second one on Schaefer's fixed point theorem [22] combined with a selection theorem due to Bressan and Colombo [1] for lower semicontinuous multivalued operators with nonempty closed and decomposable values.

Section 4 is devoted to the existence of solutions to the following Darboux problem for hyperbolic functional differential inclusions

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} \in F(x,y,z,u_{(x,y,z)}), \quad (x,y,z) \in \mathcal{D} \tag{3}$$

$$u(x,y,z) = \phi(x,y,z),$$
$$(x,y,z) \in [-r_1,a] \times [-r_2,b] \times [-r_3,c] \backslash ((0,a] \times (0,b] \times (0,c]) \tag{4}$$

where $F : \mathcal{D} \times C([-r_1,0] \times [-r_2,0] \times [-r_3,0], E) \longrightarrow \mathcal{P}(E)$ is a multivalued map, $\phi \in C([-r_1,a] \times [-r_2,b] \times [-r_3,c] \backslash ((0,a] \times (0,b] \times (0,c]), E), r_1 > 0,\ r_2 > 0, r_3 > 0$.

For each $u \in C([-r_1,a] \times [-r_2,b] \times [-r_3,c], E)$ and each $(x,y,z) \in \mathcal{D}$ the function $u_{(x,y,z)} : [-r_1,0] \times [-r_2,0] \times [-r_3,0] \to E$ is defined by

$$u_{(x,y,z)}(s,t,w) = u(x+s, y+t, z+w),$$

for each

$$(s,t,w) \in [-r_1,0] \times [-r_2,0] \times [-r_3,0].$$

Finally in Section 5 we indicate some possible generalizations of IVP (1)–(2) to nonlocal hyperbolic problems

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} \in F(x,y,z,u(x,y,z)), \quad (x,y,z) \in \mathcal{D} \tag{5}$$

$$u(x,y,0) + \sum_{k=1}^{r} \gamma_k(x,y) u(x,y,c_k) = f(x,y), \quad (x,y) \in J_a \times J_b \tag{6}$$

$$u(0,y,z) + \sum_{i=1}^{p} \upsilon_i(y,z) u(a_i,y,z) = g(y,z), \quad (y,z) \in J_b \times J_c \tag{7}$$

$$u(x,0,z) + \sum_{j=1}^{\ell} \vartheta_j(x,z) u(x,b_j,z) = h(x,z), \quad (x,z) \in J_a \times J_c, \tag{8}$$

where $F, f,\ g,\ h$ are as in the problem (1)-(2), $\gamma_k : J_a \times J_b \to E,\ \upsilon_i : J_b \times J_c \to E,\ \vartheta_j : J_a \times J_c \to E$ are given functions and a_i $(i = 1, \cdots, p)$, b_j $(j = 1, \cdots, \ell)$ and c_k $(k = 1, \cdots, r)$ are given numbers such that $0 < a_1 < \cdots < a_p \le a, \quad 0 < b_1 < \cdots < b_\ell \le b$ and $0 < c_1 < \cdots < c_r \le c$.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this paper.

$C(\mathcal{D}, E)$ denotes the Banach space of all continuous functions from $\mathcal{D}$ into E with the norm

$$\|u\|_\infty = sup\{|u(x,y,z)| : (x,y,z) \in \mathcal{D}\}.$$

$L^1(\mathcal{D}, E)$ denotes the Banach space of functions $u : \mathcal{D} \longrightarrow E$ which are Bochner integrable normed by

$$\|u\|_{L^1} = \int_0^a \int_0^b \int_0^c |u(x,y,z)| dz dy dx.$$

Let (X,d) be a metric space. We use the notations:

$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, and $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$.

Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b),\ d(a, B) = \inf_{b\in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space ([18]).

A multivalued map $F : \mathcal{D} \times E \longrightarrow P_{cl}(E)$ is said to be measurable if for each $w \in E$ the function $Y : \mathcal{D} \longrightarrow \mathbb{R}$ defined by

$$Y(x, y, z) = d(w, F(x, y, z, u)) = \inf\{d(w, v) : v \in F(x, y, z, u)\}$$

is measurable, where d is the metric introduced from the Banach space $C(\mathcal{D}, E)$.

Definition 2.1. *A multivalued operator $N : X \to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \textit{for each } x,\ y \in X,$$

b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multivalued operator N will be denoted by $FixN$.

The proof of our first result is based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [6] (see also Deimling, [7] Theorem 11.1).

Lemma 2.2. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Denote by $\mathcal{L}$ the σ-algebra of the Lebesque measurable subsets of $\mathcal{D}$ and by $\mathcal{B}(E)$ the family of all Borel subsets of E. Recall that $F : \mathcal{D} \times E \to \mathcal{P}(E)$ is called $\mathcal{L} \otimes \mathcal{B}$ measurable if for any closed subset C of E we have that $\{(x, y, z, u) \in \mathcal{D} \times E : F(x, y, z, u) \cap C \neq \emptyset\} \in \mathcal{L} \otimes \mathcal{B}$.

A subset K of $L^1(\mathcal{D},E)$ is decomposable, if for all $u,v\in K$ and $A\in\mathcal{L}$ we have $u\chi_A+v\chi_{\mathcal{D}-A}\in K$, where χ_A stands for the characteristic function of the set A.

Let E be a Banach space, X a nonempty closed subset of E and $G:X\to\mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x\in X: G(x)\cap B\neq\emptyset\}$ is open for any open set B in E. For more details on multivalued maps we refer to the books of Deimling [7], Gorniewicz [14], Hu and Papageorgiou [16] and Tolstonogov [29].

Definition 2.3. *Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1(\mathcal{D},E))$ be a multivalued operator. We say N has property (BC) if*

1) *N is lower semi-continuous (l.s.c.);*

2) *N has nonempty closed and decomposable values.*

Let $F:\mathcal{D}\times E\to\mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F}: C(\mathcal{D},E)\to\mathcal{P}(\mathcal{D},E))$$

by letting

$$\mathcal{F}(u)=\big\{w\in L^1(\mathcal{D},E)\colon w(x,y,z)\in F(x,y,z,u(x,y,z))$$

$$\text{for a.e. } (x,y,z)\in\mathcal{D}\big\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.4. *Let $F:\mathcal{D}\times E\to\mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.*

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.5. *[1]. Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1(\mathcal{D},E))$ be a multivalued operator which has property (BC). Then N has a continuous selection; i.e., there exists a continuous function (single-valued) $g:Y\to L^1(\mathcal{D},E)$ such that $g(y)\in N(y)$ for every $y\in Y$.*

3 The Darboux Problem for Hyperbolic Differential Inclusions

In this section we state and prove our first theorem for the IVP (1)–(2). First however we give the definition of a solution of the IVP (1)–(2).

Definition 3.1. *By a solution of (1)-(2) we mean a function* $u(\cdot,\cdot,\cdot) \in C(\mathcal{D}, E)$ *such that there exists* $v \in L^1(\mathcal{D}, E)$ *for which we have*

$$u(x,y,z) = Q(x,y,z) + \int_0^x \int_0^y \int_0^z v(t,s,w)dwdsdt \quad \text{for each } (x,y,z) \in \mathcal{D}$$

with $v(t,s,w) \in F(t,s,w,u(t,s,w))$ *a.e. on* $\mathcal{D}$*; here* $Q(x,y,z) = f(x,y) + g(y,z) + h(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0$.

Theorem 3.2. *Assume that:*

(H1) $F : \mathcal{D} \times E \longrightarrow P_{cp}(E)$ *has the property that* $F(\cdot,\cdot,\cdot,u) : \mathcal{D} \to P_{cp}(E)$ *is measurable for each* $u \in E$*;*

(H2) $H_d(F(t,s,w,u), F(t,s,w,\overline{u})) \le L|u - \overline{u}|$, *for each* $(t,s,w) \in \mathcal{D}$ *and* $u, \overline{u} \in E$, *where* L *is a positive constant, and*

$$H_d(0, F(t,s,w,0)) \le M(t,s,w) \quad \text{for a. e.} \quad (t,s,w) \in \mathcal{D},$$

with

$$M(\cdot,\cdot,\cdot) \in L^1(\mathcal{D}, \mathbb{R}^+).$$

Then the IVP (1)-(2) has at least one solution on $\mathcal{D}$.

Proof. Let m be a positive constant (to be chosen later) and on the space $C(\mathcal{D}, E)$ take the norm $\|\cdot\|_C$ given by

$$\|u\|_C = \sup_{(x,y,z)\in\mathcal{D}} e^{-m(x+y+z)}|u(x,y,z)|.$$

We first transform the problem (1)–(2) into a fixed point problem. Consider the multivalued operator, $N : C(\mathcal{D}, E) \to \mathcal{P}(\mathcal{D}, E))$ defined by:

$$N(u) = \{h \in C(\mathcal{D}, E) : h(x,y,z) = Q(x,y,z) +$$

$$+ \int_0^x \int_0^y \int_0^z v(t,s,w)dwdsdt, \ v \in S_{F,u}\Big\}$$

where

$$S_{F,u} = \Big\{ v \in L^1(\mathcal{D}, E) : v(t,s,w) \in F(t,s,w,u(t,s,w))$$

$$\text{for a.e. } (t,s,w) \in \mathcal{D} \Big\}.$$

Remark 3.3. *(i) It is clear that the fixed points of N are solutions to (1)-(2).*
(ii) For each $u \in C(\mathcal{D}, E)$ the set $S_{F,u}$ is nonempty, since by (H1) F has a measurable selection (see [3], Theorem III.6).

We shall show that N satisfies the assumptions of Lemma 2.2. The proof will be given in two steps.

Step 1: *$N(u) \in P_{cl}(C(\mathcal{D}, E))$ for each $u \in C(\mathcal{D}, E)$.*

Indeed, let $(h_n)_{n \geq 0} \in N(u)$ such that $h_n \longrightarrow \tilde{h}$ in $C(\mathcal{D}, E)$. Then $\tilde{h} \in C(\mathcal{D}, E)$ and there exists $g_n \in S_{F,u}$ such that for each $(x,y,z) \in \mathcal{D}$

$$h_n(x,y,z) = Q(x,y,z) + \int_0^x \int_0^y \int_0^z g_n(t,s,w) dwdsdt.$$

Using the fact that F has compact values and from (H2) we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(\mathcal{D}, E)$ and hence $g \in S_{F,u}$. Then for each $(x,y,z) \in \mathcal{D}$

$$h_n(x,y,z) \longrightarrow \tilde{h}(x,y,z) = Q(x,y,z) + \int_0^x \int_0^y \int_0^z g(t,s,w) dwdsdt,$$

so $\tilde{u} \in N(u)$.

Step 2: *$H_d(N(u_1), N(u_2)) \leq \gamma \|u_1 - u_2\|_C$ for each $u_1, u_2 \in C(\mathcal{D}, E)$ (where $\gamma < 1$).*

Let $u_1, u_2 \in C(\mathcal{D}, E)$ and $h_1 \in N(u_1)$. Then there exists $g_1(t,s,w) \in F(t,s,w,u_1(t,s,w))$ such that

$$h_1(x,y,z) = Q(x,y,z) + \int_0^x \int_0^y \int_0^z g_1(t,s,w) dwdsdt \text{ for each } (t,s,w) \in \mathcal{D}.$$

From (H2) it follows that

$$H_d(F(t,s,w,u_1(t,s,w)), F(t,s,w,u_2(t,s,w))) \leq L|u_1(t,s,w) - u_2(t,s,w)|.$$

Hence there is $p \in F(t,s,w,u_2(t,s,w))$ such that

$$|g_1(t,s,w)-p| \leq L|u_1(t,s,w)-u_2(t,s,w)|, \quad (t,s,w) \in \mathcal{D}.$$

Consider $U : \mathcal{D} \to \mathcal{P}(E)$, given by

$$U(t,s,w) = \{p \in E : |g_1(t,s,w)-p| \leq L|u_1(t,s,w)-u_2(t,s,w)|\}.$$

Since the multivalued operator $V(t,s,w) = U(t,s,w) \cap F(t,s,w,u_2(t,s,w))$ is measurable (see Proposition III.4 in [3]) there exists $g_2(t,s,w)$ a measurable selection for V. Thus $g_2(t,s,w) \in F(t,s,w,u_2(t,s,w))$ and

$$|g_1(t,s,w)-g_2(t,s,w)| \leq L|u_1(t,s,w)-u_2(t,s,w)| \text{ for each } (t,s,w) \in \mathcal{D}.$$

Let us define for each $(t,s,w) \in \mathcal{D}$

$$h_2(x,y,z) = Q(x,y,z) + \int_0^x \int_0^y \int_0^z g_2(t,s,w)dwdsdt.$$

Then we have

$$\begin{aligned}
|h_1(x,y,z)-h_2(x,y,z)| &\leq \int_0^x \int_0^y \int_0^z |g_1(t,s,w)-g_2(t,s,w)|\, dwdsdt \\
&\leq L\int_0^x \int_0^y \int_0^z |u_1(t,s,w)-u_2(t,s,w)|dwdsdt \\
&= L\int_0^x \int_0^y \int_0^z \|u_1-u_2\|_C e^{m(t+s+w)}dwdsdt \\
&\leq \frac{Le^{m(x+y+z)}}{m^3}\|u_1-u_2\|_C.
\end{aligned}$$

Thus

$$\|h_1-h_2\|_C \leq \frac{L}{m^3}\|u_1-u_2\|_C.$$

Essentially the same reasoning (obtained by interchanging the roles of u_1 and u_2) yields

$$H_d(N(u_1),N(u_2)) \leq \frac{L}{m^3}\|u_1-u_2\|_C.$$

Let m be a positive constant such that $L < m^3$. Then N is a contraction and thus, by Lemma 2.2 N has a fixed point u, which is a solution to (1)-(2).

Now Schaefer's theorem combined with a selection theorem of Bressan and Colombo for lower semicontinuous maps with nonempty closed and decomposable values also gives us an existence result for the problem (1)-(2). Before this, let us introduce the following hypotheses which are assumed hereafter:

(C1) $F : \mathcal{D} \times E \longrightarrow \mathcal{P}(E)$ is a nonempty compact valued multivalued map such that:
a) $(x, y, z, u) \mapsto F(x, y, z, u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
b) $u \mapsto F(x, y, z, u)$ is lower semi-continuous for a.e. $(x, y, z) \in \mathcal{D}$;

(C2) For each $r > 0$, there exists a function $h_r \in L^1(\mathcal{D}, \mathrm{I\!R}^+)$ such that

$$\begin{aligned} |F(x,y,z,u)| \quad &:= \quad \sup\{|v| : v \in F(x,y,z,u)\} \le h_r(x,y,z) \\ & \quad \text{for a.e. } (x,y,z) \in \mathcal{D} \text{ and } u \in E \text{ with } |u| \le r. \end{aligned}$$

In the proof of our next main result we will need the following well known theorem.

Lemma 3.4. *[13]. Let $F : \mathcal{D} \times E \to \mathcal{P}(E)$ be a multivalued map. Assume $(C1)$ and $(C2)$ hold. Then F is of l.s.c. type.*

Theorem 3.5. *Suppose, in addition to hypotheses (C1), (C2), the following also hold:*

(H3) There exist functions $p, q \in L^1(\mathcal{D}, \mathrm{I\!R}^+)$ such that

$$|F(x,y,z,u)| := sup\{|v| : v \in F(x,y,z,u)\} \le p(x,y,z) + q(x,y,z)|u|,$$

for almost all $(x, y, z) \in \mathcal{D}$ and all $u \in E$.

(H4) For each $(x, y, z) \in \mathcal{D}$, the multivalued map $F(x, y, z, \cdot)$ maps bounded sets of E into relatively compact sets of E.

Then the initial value problem (1)–(2) has at least one solution on $\mathcal{D}$.

Proof. Now (C1) and (C2) (see Lemma 3.4) that F is of lower semi-continuous type. Then from Theorem 2.5 there exists a continuous function $\mathcal{R} : C(\mathcal{D}, E) \to L^1(\mathcal{D}, E)$ such that $\mathcal{R}(u) \in \mathcal{F}(u)$ for all $u \in C(\mathcal{D}, E)$.

We consider the problem

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} = \mathcal{R}(u)(x,y,z), \quad (x,y,z) \in \mathcal{D}, \tag{9}$$

$$u(x,y,0) = f(x,y), \quad u(0,y,z) = g(y,z), \quad u(x,0,z) = h(x,z). \tag{10}$$

If $u \in C(\mathcal{D}, E)$ is a solution of the problem (9)–(10), then u is a solution to the problem (1)-(2).

We transform problem (9)–(10) into a fixed point problem by considering the operator $N : C(\mathcal{D}, E) \to C(\mathcal{D}, E)$ defined by:

$$N(u)(x, y, z) := Q(x, y, z) + \int_0^x \int_0^y \int_0^z \mathcal{R}(u)(\xi, \eta, \theta) d\theta d\eta d\xi.$$

We shall show that N is a continuous and completely continuous operator.

Step 1: N *is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \longrightarrow u$ in $C(\mathcal{D}, E)$. Then

$$|N(u_n)(x, y, z) - N(u)(x, y, z)| \leq \int_0^t \int_0^s \int_0^w |\mathcal{R}(u_n)(\xi, \eta, \theta) -$$

$$-\mathcal{R}(u)(\xi, \eta, \theta)| d\theta d\eta d\xi \leq \int_0^a \int_0^b \int_0^c |\mathcal{R}(u_n)(\xi, \eta, \theta) - \mathcal{R}(u)(\xi, \eta, \theta)| d\theta d\eta d\xi.$$

Since the function $\mathcal{R}$ is continuous, then

$$\|N(u_n) - N(u)\|_\infty \to 0 \;\; as \; n \to \infty.$$

Step 2: N *is bounded on bounded sets of* $C(\mathcal{D}, E)$.

Indeed, it is enough to show that there exists a positive constant k such that for each $u \in B_r = \{u \in C(\mathcal{D}, E) : \|u\|_\infty \leq r\}$ one has $\|N(u)\|_\infty \leq k$.

By (H2) we have

$$|(Nu)(x, y, z)| \leq |Q(x, y, z)| + \int_0^x \int_0^y \int_0^z h_r(t, s, w) dw ds dt,$$

so

$$\|N(u)\|_\infty \leq \|Q\|_\infty + \int_0^a \int_0^b \int_0^c h_r(t, s, w) dw ds dt := k.$$

Step 3: N *maps bounded sets into equicontinuous sets of* $C(\mathcal{D}, E)$.

Let $(x_1, y_1, z_1),\ (x_2, y_2, z_2) \in \mathcal{D},\ x_1 < x_2,\ y_1 < y_2,\ z_1 < z_2$. Thus we obtain

$$|(Nu)(x_2, y_2, z_2) - (Nu)(x_1, y_1, z_1)| \leq |f(x_2, y_2) - f(x_1, y_1)| +$$

$$= |g(y_2, z_2) - g(y_1, z_1)| + |h(x_2, z_2) - h(x_1, z_1)| +$$

$$+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}|\mathcal{R}(u)(t,s,w)|dwdsdt+\int_{x_1}^{x_2}\int_{0}^{y_1}\int_{z_1}^{z_2}|\mathcal{R}(u)(t,s,w)|dwdsdt+$$

$$+\int_{0}^{x_1}\int_{y_1}^{y_2}\int_{z_1}^{z_2}|\mathcal{R}(u)(t,s,w)|dwdsdt+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{0}^{z_1}|\mathcal{R}(u)(t,s,w)|dwdsdt\leq$$

$$\leq |f(x_2,y_2)-f(x_1,y_1)|+|g(y_2,z_2)-g(y_1,z_1)|+|h(x_2,z_2)-h(x_1,z_1)|+$$

$$+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}h_q(t,s,w)dwdsdt+\int_{x_1}^{x_2}\int_{0}^{y_1}\int_{z_1}^{z_2}h_q(t,s,w)dwdsdt+$$

$$+\int_{0}^{x_1}\int_{y_1}^{y_2}\int_{z_1}^{z_2}h_q(t,s,w)dwdsdt+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{0}^{z_1}|h_q(t,s,w)|dwdsdt,$$

where $q=\|u\|_\infty$. As $(x_2,y_2,z_2)\longrightarrow(x_1,y_1,z_1)$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 and (H4) together with the Arzela-Ascoli theorem we can conclude that N is completely continuous.

Step 4: *Now it remains to show that the set*

$$\Omega:=\{u\in C(\mathcal{D},E): u=\lambda N(u),\quad \textit{for some}\quad 0<\lambda<1\}$$

is bounded.

Let $u\in\Omega$. Then $u=\lambda N(u)$ for some $0<\lambda<1$ and

$$u(x,y,z)=\lambda Q(x,y,z)+\lambda\int_0^x\int_0^y\int_0^z\mathcal{R}(u(t,s,w))dwdsdt,\ \ (x,y,z)\in\mathcal{D},$$

where $\mathcal{R}$ is as described at the beginning of the proof. This implies by (H3) that for each $(x,y,z)\in\mathcal{D}$ we have

$$\begin{aligned}|u(x,y,z)| &\leq |Q(x,y,z)|+\int_0^x\int_0^y\int_0^z[p(t,s,w)+q(t,s,w)|u(t,s,w)|]dwdsdt\\ &\leq \|Q\|_\infty+\|p\|_{L^1}+\int_0^x\int_0^y\int_0^z q(t,s,w)|u(t,s,w)|dwdsdt.\end{aligned}$$

Invoking Gronwall's inequality we get that

$$|u(x,y,w)|\leq[\|Q\|_\infty+\|p\|_{L^1}]\,exp\|q\|_{L^1}:=K.$$

This shows that Ω is bounded. As a consequence of Schaefer's theorem ([22, 23]) we deduce that N has a fixed point u which is a solution to problem (9)-(10). Then u is a solution to the problem (1)-(2).

Remark 3.6. *A slight modification of the proof above (i.e. in Step 4 use the standard Leray–Schauder alternative [15]) guarantees that (H3) could be replaced by*

*(H3)** *There exist a function $q \in L^1(\mathcal{D}, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that*

$$|F(x, y, z, u)| \leq q(x, y, z)\psi(|u|),$$

for almost all $(x, y, z) \in \mathcal{D}$ and all $u \in E$ provided there exists a constant $M > 0$ with

$$\frac{M}{\|Q\|_\infty + \psi(M) \displaystyle\int_0^a \int_0^b \int_0^c q(t, s, w)dwdsdt} > 1.$$

4 The Darboux Problem for Hyperbolic Functional Differential Inclusions

Definition 4.1. *By a solution of (3)-(4) we mean a function $u(\cdot, \cdot, \cdot) \in C([-r_1, a] \times [-r_2, b] \times [-r_3, c], E)$ such that, there exists $v \in L^1(\mathcal{D}, E)$ for which we have*

$$u(x, y, z) = \phi(x, y, 0) + \phi(x, 0, z) + \phi(0, y, z) - \phi(0, 0, 0)+$$
$$+ \int_0^x \int_0^y \int_0^z v(t, s, w)dwdsdt$$

for each $(x, y, z) \in \mathcal{D}$ and $v(t, s, w) \in F(t, s, w, u_{(t,s,w)})$ a.e. on $\mathcal{D}$ and $u(x, y, z) = \phi(x, y, z)$ on $[-r_1, a] \times [-r_2, b] \times [-r_3, c] \backslash ((0, a] \times (0, b] \times (0, c])$.

Let $\bar{D} = [-r_1, a] \times [-r_2, b] \times [-r_3, c]$ and $\tilde{D} = [-r_1, a] \times [-r_2, b] \times [-r_3, c] \backslash ((0, a] \times (0, b] \times (0, c])$. The main result of this section is the following:

Theorem 4.2. *Assume that:*

(B1) *$F : \mathcal{D} \times C([-r_1, 0] \times [-r_2, 0] \times [-r_3, 0], E) \longrightarrow P_{cp}(E)$ has the property that $F(\cdot, \cdot, \cdot, u) : [-r_1, 0] \times [-r_2, 0] \times [-r_3, 0] \to P_{cp}(E)$ is measurable for each $u \in C([-r_1, 0] \times [-r_2, 0] \times [-r_3, 0], E)$;*

(B2) $H_d(F(t,s,w,u), F(t,s,w,\overline{u})) \le \overline{L}\|u-\overline{u}\|$, *for each* $(t,s,w)\in\mathcal{D}$ *and* $u,\overline{u}\in C([-r_1,0]\times[-r_2,0]\times[-r_3,0],E)$, *where* $\overline{L}$ *is a positive constant, and*

$$H_d(0, F(t,s,w,0)) \le N(t,s,w) \quad \textit{for a.e.} \quad (t,s,w)\in\mathcal{D},$$

with

$$N(\cdot,\cdot,\cdot)\in L^1(\mathcal{D},\mathbb{R}^+).$$

Then the IVP (3)-(4) has at least one solution on $[-r_1,a]\times[-r_2,b]\times[-r_3,c]$. *In (B2)* $\|\cdot\|$ *is the sup norm on* $[-r_1,0]\times[-r_2,0]\times[-r_3,0]$.

Proof. Let m be a positive constant and on the space $C(\bar{D},E)$ take the norm $\|\cdot\|_C$ given by

$$\|u\|_C = \sup_{(x,y,z)\in\bar{D}} e^{-m(x+y+z)}|u(x,y,z)|.$$

We transform the problem (3)–(4) into a fixed point problem. Consider the multivalued operator $N : C(\bar{D},E)\to\mathcal{P}(C(\bar{D},E))$ defined by:

$$N(u)=\left\{h\in C(\bar{D},E) : h(x,y,z)=\begin{cases}\phi(x,y,z), & (x,y,z)\in\tilde{D}\\ \phi(x,y,0)+\phi(x,0,z) & \\ +\phi(0,y,z)-\phi(0,0,0) & \\ +\displaystyle\int_0^x\int_0^y\int_0^z v(t,s,w)dwdsdt, & \\ (x,y,z)\in\mathcal{D} & \end{cases}\right\}$$

where

$$v\in S_{F,u}=\Big\{v\in L^1(\mathcal{D},E) : v(t,s,w)\in F(t,s,w,u_{(t,s,w)})$$

$$\text{for a.e.}\quad (t,s,w)\in\mathcal{D}\Big\}.$$

Now apply Lemma 2.2. The ideas are essentially the same as those in Section 3 so as a result we omit the details.

Also Schaefer's theorem combined with a selection theorem of Bressan and Colombo for lower semi-continuous maps guarantees our next result.

Theorem 4.3. *Suppose, in addition to hypotheses (C1), (C2), the following also hold:*

(H3)' *There exist functions* $p, q \in L^1(\mathcal{D}, \mathbb{R}^+)$ *such that*

$$|F(x,y,z,u)| := sup\{|v| : v \in F(x,y,z,u)\} \leq p(x,y,z) + q(x,y,z)\|u\|,$$

for almost all $(x,y,z) \in \mathcal{D}$ *and all* $u \in C([-r_1,0] \times [-r_2,0] \times [-r_3,0], E)$.

(H4)' *For each* $(x,y,z) \in \bar{D}$*, the multivalued map* $F(x,y,z,\cdot)$ *maps bounded sets in* $C([-r_1,0] \times [-r_2,0] \times [-r_3,0], E)$ *into relatively compact sets of* E.

Then the initial value problem (3)–(4) has at least one solution on $\mathcal{D}$.

5 Nonlocal Darboux Problem

In this section we indicate some generalizations of the problem (1)-(2). By using the same method, as in Theorem 3.2 (with obvious modifications), we can prove existence results for the nonlocal Darboux problem (5)–(8). We introduce the following additional assumptions:

(D1) $\gamma_k \in C(J_a \times J_b, E)\,(k = 1, \cdots, r), \upsilon_i \in C(J_b \times J_c, E)\,(i = 1, \cdots, p), \vartheta_j \in C(J_a \times J_c, E)\,(j = 1, \cdots, \ell), \gamma_k(x,0) = \gamma_k(0,y) = 0\,(k = 1, \cdots, r), \upsilon_i(y,0) = \upsilon_i(0,z) = 0\,(i = 1, \cdots, p), \vartheta_j(x,0) = \vartheta_j(0,z) = 0\,(j = 1, \cdots, \ell)$.

By a solution of the nonlocal problem (5)–(8) we mean a function $u(\cdot,\cdot,\cdot) \in C(\mathcal{D}, E)$ such that there exists $v \in L^1(\mathcal{D}, E)$ for which we have

$$u(x,y,z) = f(x,y) + g(y,z) + h(x,z) - \sum_{k=1}^{r} \gamma_k(x,y)u(x,y,c_k) -$$

$$- \sum_{i=1}^{p} \upsilon_i(y,z)u(a_i,y,z)$$

$$- \sum_{j=1}^{q} \vartheta_j(x,z)u(x,b_j,z) - v^1(x) - v^2(y) - v^3(z) + v^0$$

$$+ \int_0^x \int_0^y \int_0^z v(t,s,w)dwdsdt \quad \text{for each } (x,y,z) \in \mathcal{D},$$

and with $v(x,y,z) \in F(x,y,z,u(x,y,z))$ a.e. on $\mathcal{D}$.

For results on nonlocal problems the interested reader is referred to [2] and the references cited therein.

Theorem 5.1. *Assume that hypotheses (H1), (H2) and (D1) hold. Then the nonlocal problem (5)–(8) has at least one solution on $\mathcal{D}$.*

Theorem 5.2. *Assume that hypotheses (C1), (C2), (H3), (D1) hold. Then the nonlocal problem (5)–(8) has at least one solution on $\mathcal{D}$.*

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ІСНУВАННЯ РОЗВ'ЯЗКІВ ЗАДАЧІ ДАРБУ ДЛЯ ГІПЕРБОЛІЧНИХ ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ В БАНАХОВИХ ПРОСТОРАХ

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Досліджено існування розв'язків задачі Діріхле для гіперболічних диференціальних і функціональних диференціальних включень третього порядку з не опуклозначною правою частиною.