NORMAL FUNCTORS IN THE CATEGORY OF ULTRAMETRIC SPACES

©2008 p. Oleksandr SAVCHENKO

Kherson State Agrarian University

Received July 2, 2008

We show that every normal functor acting in the category of compact Hausdorff spaces **COMP** determines a functor in the category **UMET** of ultrametric spaces and nonexpanding maps. Some properties of the obtained functors in **UMET** are established. In particular, we show that these functors preserve the class of complete ultrametric spaces, this fact was already known for the hyperspace functor and the probability measure functor. We show also that every natural transformation of normal functors in **COMP** determines a natural transformation of the corresponding functors in **UMET**.

1 Introduction

Since their invention, the notion of ultrametric appeared in real analysis, number theory and topology. Later it founded numerous applications, not only in different areas of mathematics but also in physics, mathematical economics, decision theory, biology etc.

In connection with some problems of semantics of program languages, in [2] (see also [12]) an ultrametric on the set of probability measures with compact support on an ultrametric spaces is introduced. It is proved, in particular, that this construction determines a functor on the category of ultrametric spaces and nonexpanding maps. A similar construction for the so called possibility measures (idempotent measures, in another terminology) is considered in [4]. A survey of some results concerning the functors in the

MSC 2000: 54E35, 54B30, 26E30

category of ultrametric spaces is given in [8]. Note also that the construction can be also defined for the plausibility measures [1].

The aim of this paper is to extend the construction over the case of the so called normal functors in the category of compact Hausdorff spaces. The notion of normal functor is introduced by E.V. Shchepin [9]. In this note we show that every normal functor acting in the category **UMET** of compact Hausdorff spaces determines a functor in the category of ultrametric spaces and nonexpanding maps and every natural transformation of normal functors determines a natural transformation of the corresponding functors.

We investigate some properties of the obtained functors in the category **UMET**. In particular, we show that the obtained functors have continuous supports and that they preserve the class of complete ultrametric spaces.

2 Preliminaries

2.1 Ultrametric spaces and nonexpanding maps

Recall that a metric d on a set X is said to be an *ultrametric* if the following strong triangle inequality holds:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for all $x, y, z \in X$.

Recall also that a map $f\colon X \to Y$ of metric spaces (X, d) and (Y, ϱ) is called *nonexpanding* if $\varrho(f(x), f(y)) \leq d(x, y)$, for all $x, y \in X$. The ultrametric spaces and their nonexpanding maps form a category denoted by **UMET**.

2.2 Normal functors in the category of ultrametric spaces

Denote by **COMP** the category whose objects are compact Hausdorff spaces and whose morphisms are continuous maps. The notion of normal functor in the category **COMP** is introduced by E.V. Shchepin [9].

In the sequel, 'functor' means 'covariant functor'.

Definition 2.1. We say that a functor $F\colon$ **COMP** $\to$ **COMP** is *normal* if F is:

1. F preserves weight (i.e., $w(F(X)) = w(X)$, for every infinite X);

2. F is continuous;

3. F is monomorphic (i.e., F preserves embeddings);
4. F is epimorphic (i.e., F preserves the onto maps);
5. F preserves intersections;
6. F preserves preimages;
7. F preserves singletons and the empty set.

The definition above requires some comments. Continuity of a functor F means that it commutes with the limits of inverse systems over directed sets.

For a monomorphic functor F and any closed subset A of a compact Hausdorff space X, we identify $F(A)$ with the subset $F(i)(F(A))$ of $F(X)$, where $i\colon A \to X$ denotes the inclusion map. That F preserves the intersections means that $F(\cap_{\alpha\in\Gamma} A_\alpha) = \cap_{\alpha\in\Gamma} F(A_\alpha)$, for every family $\{A_\alpha \mid \alpha \in \Gamma\}$ of closed subsets of a compact Hausdorff space X. Given a monomorphic functor F that preserves the intersections, for any $a \in F(X)$, we define the *support* $\operatorname{supp}(a)$ as $\cap\{A \mid A$ is a closed subset of X and $a \in F(X)\}$. By $F_\omega(X)$ we denote the set of points of finite support in the set $F(X)$.

The preservation of preimages means that, for any map $f\colon X \to Y$ in **COMP** and any closed subset B of Y, we have $F(f)^{-1}(F(B)) = F(f^{-1}(F(B)))$.

A functor F is called *almost normal* (respectively *weakly normal*) if it preserves all the properties from the definition of normal functor but the preimage-preservation (respectively of being epimorphic).

2.3 Extension of normal functors onto the category of Tychonov spaces

By β we denote the Stone-Čech compactification functor acting from the category **COMP** to the category **TYCH** of Tychonov spaces and continuous maps.

The following construction is described by Chigogidze [3]. Given a normal functor $F\colon \mathbf{COMP} \to \mathbf{COMP}$ and a Tychonov space X, we let

$$F_\beta(X) = \{a \in F(\beta X) \mid \operatorname{supp}(a) \subset X \subset \beta X\}.$$

If $f\colon X \to Y$ is a morphism in **TYCH**, then $F(\beta(f))(F_\beta(X)) \subset F_\beta(Y)$ and we denote by $F_\beta(f)$ the restriction $F(\beta(f))|F_\beta(X)\colon F_\beta(X) \subset F_\beta(Y)$. The obtained functor F_β in the category **TYCH** is a normal functor in the sense of [3].

For the sake of notational simplicity, we keep the notation F for the extended functor over the category **TYCH**.

3 Main results

Let (X, d) be au ultrametric space. For any $r > 0$ denote by $\mathcal{D}_r$ the family of decompositions D of X satisfying the conditions: every element of D is a union of balls of radii $\geq r$. We denote by $\mathcal{F}_r$ the family of quotient maps $\{X \to X/D \mid D \in \mathcal{D}_r\}$.

Define the function $\hat{d}\colon F(X) \times F(X) \to \mathbb{R}$ as follows:

$$\hat{d}(a, b) = \inf\{r > 0 \mid F(f)(a) = F(f)(b), \text{ for every } f \in \mathcal{F}_r\}.$$

We first remark that this function is well-defined. Indeed, let $a, b \in F(X)$. Then the set $A = \operatorname{supp}(a) \cup \operatorname{supp}(b)$ is compact and therefore bounded. Suppose that $\operatorname{diam}(A) < r_0$, for some $r_0 > 0$. Then, for any $f \in \mathcal{F}_{r_0}$, the set $f(A)$ is a singleton and therefore $F(f)(a) = F(f)(b)$.

Theorem 3.1. *The function $\hat{d}$ is an ultrametric on the set $F(X)$.*

Proof. It is obvious that $\hat{d}(a, b) \geq 0$, for all $a, b \in F(X)$. Suppose now that $a \neq b$. Since the set of points with finite supports is dense in $F(X)$ (see [9]), there exists $r_0 > 0$ and $f \in \mathcal{F}_{r_0}$ such that $F(f)(a) \neq F(f)(b)$. We therefore conclude that $\hat{d}(a, b) \geq r_0 > 0$.

It is clear that the function $\hat{d}$ is symmetric.

We are going to show that $\hat{d}$ satisfies the strong triangle inequality. Let $a, b, c \in F(X)$ and suppose that $\hat{d}(a, b) < r$, $\hat{d}(b, c) < r$, for some $r > 0$. Then there exist $D_1, D_2 \in \mathcal{D}_r$ such that, for the quotient maps $q_i\colon X \to X/D_i$, $i = 1, 2$, we have $F(q_1)(a) = f(q_1)(b)$ and $F(q_2)(b) = F(q_2)(c)$. Denote by D the decomposition

$$D_1 \wedge D_2 = \{U_1 \cap U_2 \mid U_i \in D_i,\ i = 1, 2\} \setminus \{\emptyset\},$$

then from the properties of ultrametrics it easily follows that $D \in \mathcal{D}_r$. The quotient map $q\colon X \to X/D$ has the property that $q = f_1 q_1 = f_2 q_2$, for some f_1, f_2. Then

$$F(q)(a) = F(f_1 q_1)(a) = F(f_1 q_1)(b) = F(f_2 q_2)(b) = F(f_2 q_2)(c) = F(q)(c),$$

whence $\hat{d}(a, c) < r$ and the strong triangle inequality holds. □

In the sequel, we endow the spaces of the form $F(X)$, for an ultrametric space (X, d) with the ultrametric $\hat{d}$ defined above.

Theorem 3.2. *Let (X, d), (Y, ϱ) be ultrametric spaces and let $f\colon X \to Y$ be a nonexpanding map. Then the map $F(f)\colon F(X) \to F(Y)$ is also nonexpanding.*

Proof. Let $a, b \in F(X)$ and $\hat{\varrho}(F(f)(a), F(f)(b)) < r$, for some $r > 0$. Then for every $g \in \mathcal{F}_r$ we have $F(g)(F(f)(a)) = F(g)(F(f)(b))$. Since the map f is nonexpanding, we conclude that $gf \in \mathcal{F}_r$, whence $\hat{d}(a, b) < r$. This completes the proof. □

We therefore obtain an endofunctor in the category **UMET** (we keep the notation F for such a functor).

Let $\varphi = (\varphi_X) \colon F \to G$ be a natural transformation of normal functors in **COMP**. It is known that such a transformation uniquely determines the natural transformation of the extended functors over **TYCH** (for which we keep the same notation).

Theorem 3.3. *Any natural transformation $\varphi = (\varphi_X) \colon F \to G$ of normal functors in* **COMP** *determines a natural transformation of the correspondi-ng functors in* **UMET**.

Proof. Let (X, d) be an ultrametric space, $a, b \in F(X)$ and $\hat{d}(a, b) < r$. Then, for every $f \colon X \to Y$, $f \in \mathcal{F}_r$, we have $F(f)(a) = F(f)(b)$. Let ϱ denote the ultrametric on Y. From the commutativity of the diagram

$$\begin{array}{ccccc} & & F(f) & & \\ & F(X) & \longrightarrow & F(Y) & \\ \varphi_X & \downarrow & & \downarrow & \varphi_Y \\ & G(X) & \longrightarrow & G(Y) & \\ & & G(f) & & \end{array}$$

we conclude that

$$G(f)\varphi_X(a) = \varphi_Y F(f)(a) = \varphi_Y F(f)(b) = G(f)\varphi_X(b),$$

whence $\hat{d}(\varphi_X(a), \varphi_X(a)) < r$. We therefore conclude that the maps φ_X are nonexpanding. □

Proposition 3.4. *The set $F_\omega(X)$ is dense in the space $F(X)$, for any ultrametric space (X, d) and any normal functor F.*

Proof. Let $a \in F(X)$ and $\varepsilon > 0$. Let $D \in \mathcal{D}_\varepsilon$. For any element $C \in D$, choose a point $x_C \in C$. It is clear that the set $Y = \{x_C \mid C \in D\}$ is closed in X, being a discrete space in the sense that $d(x, y) \geq \varepsilon$, for every $x, y \in Y$, $x \neq y$.

Denote by $f \colon X \to Y$ the retraction such that $f(C) = \{x_C\}$, for every $C \in D$. Then $F(f)(a) \in F_\omega(X)$, because $\mathrm{supp}(F(f)(a)) = f(\mathrm{supp}(a))$ is a compact subset of Y and therefore finite. One can easily see that $\hat{d}(a, F(f)(a)) \leq \varepsilon$. □

4 Examples

Below we list some examples of normal functors and describe the ultrametric.

1) We start with the trivial example of the identity functor. Obviously, the above construction determines also the identity functor on the category **UMET**.

2) Let $F = (-)^n$, where $1 < n < \omega$ (the power functor). One can easily see that, for an ultrametric space (X, d), the metric $\hat{d}$ looks as follows:

$$\hat{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max\{d(x_i, y_i) \mid i = 1, \dots, n\}$$

(the l_∞-metric).

3) Let $F = (-)^\omega$ (the countable power functor). In this case one also obtains the l_∞-metric on X^ω. Note that this metric does not generate the product topology on X^ω, for compact X.

4) Let exp denote the hyperspace functor. Recall that the hyperspace $\exp(X)$ of a Tychonov space X is the set of all nonempty compact subsets of X. The Vietoris topology on $\exp(X)$ is the topology whose base consists of the sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp(X) \mid A \subseteq \cup_{i=1}^n A_i,\ A \cap U_i \neq \emptyset \text{ for every } i\},$$

where $U_1, \dots, U_n$ run over the topology of X, $n \in \mathbb{N}$. If (X, d) is a metric space, then the topology of $\exp(X)$ is that generated by the Hausdorff metric d_H:

$$d_H(A, B) = \inf\{r > 0 \mid A \subset O_r(B),\ B \subset O_r(A)\}$$

(here $O_t(C)$ stands for the r-neighborhood of C). It is known (and easy to see) that the Hausdorff metric d_H is an ultrametric if so is d.

We are going to show that, for any ultrametric space (X, d), the ultrametric $\hat{d}$ coincides with d_H. Let $A, B \in \exp(X)$. If $d_H(A, B) < r$, then every ball $B_r(x)$ either intersects both A and B or misses both A and B. Then, for every $f \in \mathcal{F}_r$, we have $f(A) = f(B)$, whence $\hat{d}(A, B) < r$ and we conclude that $\hat{d} \leq d_H$. On the other hand, if $\hat{d}(A, B) < r$, then consider the decomposition $D = \{B_r(x) \mid x \in X\}$ and let $f\colon X \to X/D$ be the quotient map; then $f(A) = f(B)$, whence $A \subset O_r(B)$, $B \subset O_r(A)$ and we conclude that $d_H(A, B) < r$.

5) The functor P of probability measures. The set $P(X)$ consists of probability measures with compact support on X. Hartog and Vink [2] introduced the ultrametric on the set $P(X)$ as follows. For $r > 0$, let $\mathcal{O}_r$ consist of all sets that can be represented as the unions of balls of radii r.

Since, for every $D \in \mathcal{D}_r$, the elements of D belong to $\mathcal{O}_r$, we conclude that the ultrametric described above coincides with that introduced by Hartog and Vink.

6) One can similarly prove that, for the functor I of idempotent measures, the metric defined above coincides with that defined by Hubal' and Zarichnyi [4].

4.1 Properties

In the sequel, F is a normal functor in the category **COMP**.

Proposition 4.1. *Let X be a finite ultrametric space. The ultrametric space $F(X)$ is discrete in the sense that there is $c > 0$ such that, for any distinct $a, b \in F(X)$, we have $\hat{d}(a, b) \geq c$.*

Proof. Let $c = \min\{d(x, y) \mid x, y \in X,\ x \neq y\}$. Then $c > 0$. If $a, b \in F(X)$, $a \neq b$, and $F(f)(a) = F(f)(b)$, for some $f \in \mathcal{F}_r$, $r > 0$, then necessarily $r \geq c$, whence the result follows. □

Theorem 4.2. *The map* $\mathrm{supp}\colon F(X) \to \exp(X)$ *is nonexpanding.*

Proof. Let $a, b \in F(X)$ and $d_H(\mathrm{supp}(a), \mathrm{supp}(b)) \geq r$. Then, for any $r' < r$ and any $f \in \mathcal{D}_{r'}$, we have $f(\mathrm{supp}(a)) \neq f(\mathrm{supp}(b))$, whence $F(f)(a) \neq F(f)(b)$, and therefore $\hat{d}(a, b) \geq r'$. Since $r' < r$ is arbitrary, we conclude that $\hat{d}(a, b) \geq d_H(\mathrm{supp}(a), \mathrm{supp}(b))$. Thus the map supp is nonexpanding. □

It follows from Theorem 4.2 that the map supp is continuous and therefore all the functors in the category **UMET** generated by normal functors in **COMP** are functors with continuous supports. Note that this is not the case for the functors in the category **COMP**.

Remark also that, for a compact ultrametric space (X, d), the topology on $F(X)$ induced by the metric $\hat{d}$, does not necessarily coincide with the initial topology of $F(X)$. Indeed, this easily follows from the existence of the normal functors with discontinuous supports (e.g., the functor P of probability measures).

The following example demonstrates that, even for the functors of finite degree with continuous supports, the topology on $F(X)$ induced by the metric $\hat{d}$, does not necessarily coincide with the initial topology of $F(X)$. Indeed, denote by P_2 the functor of probability measures with support of cardinality at most 2. The elements of $P_2(X)$ are of the form $\mu = t\delta_x + (1 - t)\delta_y$, where $x, y \in X$ and $t \in [0, 1]$. We define the subfunctor F of P_2 as follows:

$$F(X) = \{t\delta_x + (1 - t)\delta_y \in P_2(X) \mid t \in \{0, 1\} \cup [1/3, 2/3]\}.$$

It is easy to see that

$$\hat{d}(t\delta_x + (1-t)\delta_y, s\delta_x + (1-s)\delta_y) = d(x,y),$$

for any distinct $s, t \in [1/3, 2/3]$.

Proposition 4.3. *Let X be a closed subset of an ultrametric space Y. Then $F(X)$ is a closed subset of $F(Y)$.*

Proof. It is easy to see that $F(X) = \mathrm{supp}^{-1}(\exp(X))$ and the result follows from Theorem 4.2 and the fact that $\exp(X)$ is a closed subset of $\exp(Y)$. □

Proposition 4.4. *Let (X,d) be a bounded ultrametric space. Then* $\mathrm{diam}(X,d) = \mathrm{diam}(F(X), \hat{d})$.

Proof. The result follows from the fact that, for the constant map $f\colon X \to \{*\}$, we have $F(f)(a) = F(f)(b)$, for every $a, b \in F(X)$, because F preserves the singletons. □

Theorem 4.5. *Let (X,d) be a complete ultrametric space. Then the space $(F(X), \hat{d})$ is also complete.*

Proof. Let $(a_i)_{i=1}^\infty$ be a Cauchy sequence in $F(X)$. Since the space $\exp(X)$ is complete (see [13]), the sequence $(\mathrm{supp}(a_i))_{i=1}^\infty$ is convergent. Let us denote its limit by A.

For any $n \in \mathbb{N}$, let us choose a finite disjoint family $\mathcal{B}_n$ of balls of radii $1/n$ that covers A and such that every element of this family intersects A. Let A_n be a subset of A such that every element of $\mathcal{B}_n$ contains precisely one of the points of A_n. Denote by $r_n\colon \cup\mathcal{B}_n \to A$ the retraction that sends every element of $\mathcal{B}_n$ into the point of A_n belonging to this element.

Then the sequence $(F(r_n)(a_i))_{i=1}^\infty$ is stationary; this easily follows from the fact that this sequence is convergent and Proposition 4.1.

let us denote its limit by b_n, $n \in \mathbb{N}$. Obviously, $\mathrm{supp}(b_n) \subset A$, for every $n \in \mathbb{N}$. There exist maps $s_n\colon \mathrm{supp}(b_{n+1}) \to \mathrm{supp}(b_n)$ such that $s_n r_{n+1} = r_n$, for every $n \in \mathbb{N}$. One can easily see that $F(s_n)(b_{n+1}) = b_n$.

Because of the completeness of X, one can naturally identify $A = \varprojlim\{\mathrm{supp}(b_n), s_n\}$ with a subset of X. The sequence $(b_n)_{n=1}^\infty$ is therefore convergent in the space $F(A)$ (endowed with the initial topology); let us denote its limit by a. We are going to show that a is the limit of the sequence $(a_i)_{i=1}^\infty$ in the space $(F(X), \hat{d})$.

Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Note that $\hat{d}(a, b_n) \le 1/n$. There exists $N \in \mathbb{N}$ such that $\hat{d}(b_n, a_m) < 1/n$, for all $m > N$. Then $\hat{d}(a, a_m) < 1/n$, for all $m > N$, whence the assertion follows.

□

We note that the corresponding results for the functor of probability measures is proved in [2].

5 Remarks and open questions

Question 5.1. A metric space (X, d) is said to be *uniformly disconnected* [6, 7] if, there exists $c \in (0, 1)$ such that, for every natural n and every $x_0, x_1, \ldots, x_n \in X$, we have

$$cd(x_0, x_n) \leq \max\{d(x_{i-1}, x_i) \mid i = 1, \ldots, n\}.$$

The uniformly disconnected spaces and Lipschitz maps form a category. A natural question arises of extension of the mentioned results over this category.

Question 5.2. In [10], metric spaces with subdominant ultrametric are considered and characterized. We leave as an open problem whether there exists a natural metric on the spaces of the form $F(X)$, for the mentioned spaces.

A normal functor F is called finitary if F preserves the class of finite sets. We conjecture that, for a finitary functor F with finite support, the topology on $F(X)$ induced by the ultrametric $\hat{d}$ coincides with the initial topology on $F(X)$, for any ultrametric space (X, d).

[1] Friedman, N. and J. Y. Halpern (1995). Plausibility measures: a user's guide. In: Proc. Eleventh Conference on Uncertainty in Artificial Intelligence (UAI '95), pp. 175–184.

[2] J.I. den Hartog, E.P. de Vink, *Building metric structures with the Meas functor*, preprint. Heinonen, J. Lectures on analysis on metric spaces. Springer-Verlag, New York, 2001.

[3] A.Ch. Chigogidze, *On extension of normal functors.* Vestn. Mosk. Univ. - 1984. - **6**. - P. 23–26.

[4] O. Hubal, M. Zarichnyi, *Idempotent probability measures on ultrametric spaces*, Journal of Mathematical Analysis and Applications, Volume 343, Issue 2, 2008, 1052-1060.

[5] E.P. de Vink and J.J.M.M. Rutten, *Bisimulation for probabilistic transition systems: a coalgebraic approach.* Theoretical Computer Science, 221(1/2):271–293, June 1999.

[6] G. David, S. Semmes, Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure. - Oxford Lecture Ser. Math. Appl. 7, Oxford University Press, 1997.

[7] Tyson, J. T.; Wu, Jang-Mei, *Characterizations of snowflake metric spaces.* Ann. Acad. Sci. Fenn. Math. 30 (2005), no. 2, 313–336.

[8] Zarichnyi M. *Spaces of measures related to dequantization.* J. Phys. Stud. 11 (2007), no. 1, 34–40.

[9] E.V. Shchepin, *Functors and uncountable powers of compacta*, Uspekhi mat. nauk, **36**(1983), no.3, 3–62.

[10] A.J. Lemin, *On ultrametrization of general metric spaces.* Proc. Amer. Math. Soc. 131 (2003), no. 3, 979–989.

[11] F. van Breugel, S. Watson, *A Note on Hyperspaces and Terminal Coalgebras*, Electronic Notes in Theoretical Computer Science Volume 19, 1999, 201-208. CMCS'99, Coalgebraic Methods in Computer Science.

[12] F. van Breugel, J. Worrell, *Towards Quantitative Verification of Probabilistic Transition Systems.* - In: Lecture Notes in Computer Science, Springer Berlin / Heidelberg, 1611-3349 (Online) Volume 2076/2001 Automata, Languages and Programming, 2001, 421-432.

[13] J.W. de Bakker, J.I. Zucker, *Processes and denotational semantics of concurrency*, Information and Control, **54**(1982), 70–120.

НОРМАЛЬНІ ФУНКТОРИ В КАТЕГОРІЇ УЛЬТРАМЕТРИЧНИХ ПРОСТОРІВ

Олександр САВЧЕНКО

Херсонський державний аграрний університет

Ми показуємо, що кожен нормальний функтор, який діє в категорії компактних гаусдорфових просторів **COMP**, визначає функтор в категорії **UMET** ультраметричних просторів та нерозтягуючих відображень. Ми встановлюємо деякі властивості одержаних при цьому функторів. Зокрема, показуємо, що одержані функтори зберігають клас повних ультраметричних просторів - цей факт був раніше відомий для функтора гіперпростору та функтора ймовірнісних мір. Показуємо також, що кожне природне перетворення нормальних функторів в **COMP** визначає природне перетворення відповідних функторів у категорії **UMET**.