INVESTIGATION OF SPECTRUM AND BOUND STATES OF THE ENERGY OPERATOR OF TWO-MAGNON SYSTEM IN A NON-HEISENBERG FERROMAGNETIC WITH SPIN $S = 2$ AND NEAREST-NEIGHBOR INTERACTIONS

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We consider a two-magnon systems in an $\nu-$ dimensional isotropic Non-Heisenberg ferromagnet with spin value $S = 2$ and nearest- neighbor interactions. Spectrum and bound states (BS) of the system for all values of full quasi-momentum Λ, and for arbitrary value of lattice dimensionality ν, and for all values of Hamiltonian parameters are investigated. We show that (i) for arbitrary $\nu \geq 2$ and for full quasi-momentum in the form $\Lambda = (\Lambda_1; \Lambda_2; ...; \Lambda_\nu) = (\Lambda_0; \Lambda_0; ...; \Lambda_0)$ the change of energy spectrum of the system is similar to that observed in the case of $\nu = 1$. In this case the operator $\widetilde{H}_2$ with $J + 5J_1 - 83J_2 + 773J_3 \neq 0$ has only one additional BS. (ii) The energy z of this additional BS is degenerate $\nu - 1$ times. (iii) If $\Lambda \neq (\Lambda_0; \Lambda_0; ...; \Lambda_0)$, we show the existence no more $2\nu + 1$ BS in the system in $\nu-$dimensional lattice.

Two-magnon systems have attracted the attention of many researchers. Probably, such systems were first discussed by Bethe [1] in the context of one-dimensional integer-valued lattices. Bethe proved that no more than one bound state (BS) of the system can exist in the case of one dimensional isotropic ferromagnet. Worts [2] examined the two-magnon system in a

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$d-$dimensional integer-valued lattice for an arbitrary d and proved that in this case, the system has $0, 1, 2, ..., d$ BSs.

Majumdar [3] investigated the two-magnon system in a one-dimensional Heisenberg ferromagnet with a coupling between nearest and second nearest neighbors for the full quasi momentum $\Lambda = \pi$. He found the spectrum and the BSs of the system numerically. In [4], such a system was examined for the case of a one- dimensional Heisenberg isotropic ferromagnet with a nearest- and second nearest-neighbor interactions for $\Lambda = \pi$ and $\Lambda = \frac{\pi}{2}$. The spectrum and the BSs of the system for these values of Λ were studied with numerical methods. Gochev [5] considered the two-magnon system in a one-dimensional Heisenberg longitudinal ferromagnet with a coupling between nearest and second nearest neighbors for an arbitrary full quasi momentum. He investigated the spectrum and the BSs of the system analytically.

The two-magnon systems in the anisotropic Heisenberg model with a nearest-neighbor interaction were addressed in [6]. The focus in [7] was on two-magnon systems in a one-dimensional anisotropic Heisenberg ferromagnet with a interaction between nearest and second nearest neighbors. The spectrum and the BSs of such systems were investigated for all values of the full quasi momentum.

The usual starting point for theoretical studies of magnetically organized matter is the Heisenberg exchange Hamiltonian (with an arbitrary spin s)

$$H = J \sum_{m,\tau} (\vec{S}_m \vec{S}_{m+\tau}), \tag{1}$$

where J is the bilinear exchange interaction parameter for nearest-neighbor atoms, $\vec{S}_m = (S_m^x; S_m^y; S_m^z)$ is the atomic spin operator of the m th node of the $\nu-$ dimensional integer-valued lattice Z^ν, and τ denotes summation over the nearest neighbors. However, the actual isotropic spin exchange Hamiltonian with an arbitrary spin s has the form [8]

$$H = \sum_{m,\tau} \sum_{n=1}^{2s} J_n (\vec{S}_m \vec{S}_{m+\tau})^n, \tag{2}$$

where J_n are the multipolar exchange interaction parameters for nearest-neighbors atoms. Hamiltonian (2) coincides with Hamiltonian (1) only for $s = 1/2$, while there exist terms with higher powers of $\vec{S}_m \vec{S}_{m+\tau}$ up to $(\vec{S}_m \vec{S}_{m+\tau})^{2s}$ inclusive for $s > 1/2$. These terms be taken into account. Hamiltonian (2) is called the non-Heisenberg Hamiltonian.

Spectrum and BSs of two-magnon system in the non-Heisenberg ferromagnet with the bilinear and biquadratic exchange interactions were studied in works [9-16]. The spectrum and the BSs of two-magnon systems in a non-Heisenberg ferromagnet with coupling between nearest neighbors by bilinear and biquadratic interactions were investigated in [9-13]. Different methods, such as the Green's function method, the molecular field approximation method, the random phase approximation method, numerical methods, and the use of the creation and annihilation operators through the Holsten-Primakoff transformation, Dyson transformation, Dyson-Maleev transformation, Golghirch transformation, and others, were applied in these works. In [14-15] , the spectrum and the BSs of this system were investigated for the case of a one-dimensional non-Heisenberg ferromagnet with $s = 1$ and with a coupling between second nearest and third nearest neighbors respectively. The values of the Hamiltonian parameters for which the BSs exist were found, and the energies of these BSs were calculated. In [16], the spectrum and the BSs of two-magnon system were investigated in a ν-dimensional non-Heisenberg ferromagnet with $s = 1$ and with a coupling between nearest neighbors.

The spectrum and the BSs of two-magnon systems in a non-Heisenberg ferromagnet with coupling between nearest neighbors by bilinear and biquadratic and quadrupolar interactions were investigated in [17-18].

In the present work we considered two-magnon system in a $\nu-$ dimensional integer-valued lattice Z^ν with nearest- neighbor interaction with the bilinear and biquadratic and quadrupolar and octupolar exchange couplings, i. e. in a $\nu-$ dimensional isotropic non-Heisenberg ferromagnet with spin value $s = 2$. We is described the change of the energy spectrum of the system in the one-dimensional lattice for all values of Hamiltonian parameters and for arbitrary value of full quasi-momentum (see Theorem 4-5). We show that for arbitrary $\nu \geq 2$ and for full quasi-momentum Λ in the form $\Lambda = (\Lambda_0; \Lambda_0; ...; \Lambda_0) \in T^\nu$ the change of the energy spectrum of the system is similar to that investigated in the case of $\nu = 1$. Only one additional BS Ψ appears, whose energy value z. This energy level is degenerate $\nu - 1$ times. For all other values of the full quasi momentum Λ of the system the operator $\tilde{H}_2$ has no more than $2\nu + 1$ BSs (taking the energy degeneration order into account) with the energy values lying outside the continuous spectrum of the system.

The methods of this study differ by their simplicity and generality . This investigation is based on finding zeros of the Fredholm determinant of the Hamiltonian.

The system Hamiltonian

$$H = -J\sum_{m,\tau}(\vec{S}_m\vec{S}_{m+\tau}) - J_1\sum_{m,\tau}(\vec{S}_m\vec{S}_{m+\tau})^2 -$$

$$-J_2\sum_{m,\tau}(\vec{S}_m\vec{S}_{m+\tau})^3 - J_3\sum_{m,\tau}(\vec{S}_m\vec{S}_{m+\tau})^4, \qquad (3)$$

acts in the symmetrical Fock's space $\mathcal{H}$. Here $\vec{S}_m$ is the atomic spin $s = 2$ operator in the node m, $J > 0$ and $J_1 > 0$ and $J_2 > 0$ and $J_3 > 0$ are the bilinear and biquadratic and quadrupolar and octupole interaction parameters for nearest- neighbor atoms of the lattice, and τ denotes summation over the nearest neighbors. We set $S_m^{\pm} = S_m^x \pm iS_m^y$. Let φ_0 be the so-called vacuum vector, which is fully determined by the conditions $S_m^+\varphi_0 = 0$ and $S_m^z\varphi_0 = 2\varphi_0, ||\varphi_0|| = 1$. The vectors $S_m^-S_n^-\varphi_0$ describe the state of the system of two magnons located at the nodes m and n. Let $\mathcal{H}_2$ be the closure of the space formed by all linear combinations of these two vectors. This space is called the two-magnon space of the operator H.

Proposition 1. *The space $\mathcal{H}_2$ is invariant with respect to the operator H. The operator $H_2 = H/_{\mathcal{H}_2}$ is a bounded self-adjoint operator. It generates the bounded self-adjoint operators $\overline{H_2}$, acting in the space $l_2((Z^\nu)^2)$ according to the formula*

$$(\overline{H_2}f)(p;q) = -J\sum_{p;q;\tau}\{(\delta_{p,q+\tau}+\delta_{p+\tau,q}-8)f(p;q)+1/2(4-\delta_{p,q+\tau})f(p-\tau;q)+$$

$$+1/2(4-\delta_{p+\tau,q})f(p;q-\tau)+1/2(4-\delta_{p+\tau,q})f(p+\tau,q)+$$

$$+1/2(4-\delta_{p,q+\tau})f(p;q+\tau)\} - J_1\sum_{p;q;\tau}\{(-\delta_{p,q+\tau}-\delta_{p+\tau,q}+32+12\delta_{p,q})f(p;q)-$$

$$-1/2(5\delta_{p,q+\tau}+12\delta_{p,q}+16)f(p-\tau,q)-1/2(5\delta_{p+\tau,q}+12\delta_{p,q}+16)f(p+\tau,q)-$$

$$-1/2(5\delta_{p,q+\tau}+12\delta_{p,q}+16)f(p;q+\tau)-1/2(5\delta_{p+\tau,q}+12\delta_{p,q}+16)f(p;q-\tau)+$$

$$+6\delta_{p,q}f(p-\tau;q-\tau)+6\delta_{p,q+\tau}f(p-\tau;q+\tau)+6\delta_{p+\tau,q}f(p+\tau;q-\tau)+$$

$$+6\delta_{p,q}f(p+\tau;q+\tau)\} - J_2\sum_{p;q;\tau}\{(-23\delta_{p,q+\tau}-23\delta_{p+\tau,q}-128-132\delta_{p,q})f(p;q)+$$

$$+1/2(83\delta_{p,q+\tau}+132\delta_{p,q}+64)f(p;q+\tau)+1/2(83\delta_{p+\tau,q}+132\delta_{p,q}+64)f(p+\tau;q)+$$

$$+1/2(83\delta_{p,q+\tau}+132\delta_{p,q}+64)f(p-\tau,q)+1/2(83\delta_{p+\tau,q}+132\delta_{p,q}+64)f(p;q-\tau)-$$

$$-66\delta_{p,q}f(p-\tau;q-\tau)-66\delta_{p,q}f(p+\tau;q+\tau)-60\delta_{p,q+\tau}f(p-\tau;q+\tau)-$$

$$-60\delta_{p+\tau,q}f(p+\tau;q-\tau)\}-J_3\sum_{p;q;\tau}\{(295\delta_{p,q+\tau}+295\delta_{p+\tau,q}+512+1116\delta_{p,q})f(p;q)-$$

$$-1/2(773\delta_{p,q+\tau}+1116\delta_{p,q}+256)f(p;q+\tau)-1/2(773\delta_{p+\tau,q}+1116\delta_{p,q}+256)\times$$

$$\times f(p+\tau;q)-1/2(773\delta_{p+\tau,q}+1116\delta_{p,q}+256)f(p;q-\tau)+$$

$$+558\delta_{p,q}f(p-\tau;q-\tau)+558\delta_{p,q}f(p+\tau;q+\tau)+$$

$$+474\delta_{p,q+\tau}f(p-\tau;q+\tau)+474\delta_{p+\tau,q}f(p+\tau;q-\tau)\}, \tag{4}$$

where $\delta_{k,j}$ is the Kronecker symbol. The operator H_2 acts on the vector $\Psi \in \mathcal{H}_2$ according to the formula

$$H_2\Psi=\sum_{p,q}(\overline{H}_2 f)(p;q)S_p^- S_q^- \varphi_0. \tag{5}$$

Proof. The proof is by direct calculation in which we use the well-known commutation relations between the operators S_m^+, S_p^-, and S_m^z. □

Lemma 1. *The spectra of the operators H_2 and $\overline{H_2}$ coincide.*

Proof. Because H_2 and $\overline{H}_2$ are bounded self-adjoint operators, it follows that if $\lambda \in \sigma(H_2)$, then the Weyl criterion (see [20]) implies that there is sequence $\{\Psi_n\}_{n=1}^{\infty}$ such that $||\Psi_n|| = 1$ and $\lim_{n\to\infty} ||(H_2-\lambda)\Psi_n|| = 0$. We set $\Psi_n = \frac{1}{\sqrt{16+8\delta_{p,q}}}\sum_{p,q} f_n(p;q)S_p^- S_q^- \varphi_0$. Then $||(H_2-\lambda)\Psi_n||^2 = ((H_2-\lambda)\Psi_n,(H_2-\lambda)\Psi_n) = \sum_{p,q}|(\overline{H}_2 f_n)(p;q)-\lambda f_n(p;q)|^2 == ||\overline{H}_2 F_n-\lambda F_n||^2 \to 0$ as $n\to\infty$, where $F_n = \sum_{p,q} f_n(p;q)$. It follows that $\lambda \in \sigma(\overline{H}_2)$. Consequently, $\sigma(H_2) \subset \sigma(\overline{H}_2)$. Then, by the Weyl criterion, there is a sequence $\{F_n\}_{n=1}^{\infty}$ such that $||F_n|| = 1$ and $\lim_{n\to\infty}||(\overline{H}_2-\overline{\lambda})F_n|| = 0$. Setting $F_n = \sum_{p,q} f_n(p;q)$, $||F_n|| = (\sum_{p,q}|f_n(p;q)|^2)^{1/2}$, we conclude that $||\Psi_n|| = ||F_n|| = 1$ and $||(\overline{H}_2-\overline{\lambda})F_n|| = ||(H_2-\overline{\lambda})\Psi_n|| \to 0$ as $n\to\infty$. This means that $\overline{\lambda} \in \sigma(H_2)$ and hence $\sigma(\overline{H}_2) \subset \sigma(H_2)$. These two relations imply $\sigma(H_2) = \sigma(\overline{H}_2)$. □

As is seen, if vector $\Psi = \sum_{p,q} f(p;q)S_p^- S_q^- \varphi_0$ is an eigenfunction of H_2 with the eigenvalue $z \notin G_\nu$, then $F = \sum_{p,q} f(p;q)$ is an eigenfunction of the operator $\overline{H}_2$ with the same eigenvalue $z \notin G_\nu$, and this eigenvalue has the same multiplicity. Therefore, to investigate the spectrum of the operator H_2, it suffices to consider that of the operator $\overline{H}_2$ acting in $l_2(Z^\nu \times Z^\nu)$ by formula (4).

The objective of this work is to investigate the spectrum and the BSs of the operator $\overline{H_2}$, for which the momentum representation is convenient. Let $\mathcal{F}$ be the Fourier transformation

$$\mathcal{F} : l_2((Z^\nu)^2) \Rightarrow L_2((T^\nu)^2) \equiv \tilde{\mathcal{H}}_2,$$

where T^ν is a $\nu-$ dimensional cube, $T = [0; 2\pi)$ is a interval with the normalized Lebesgue measure $d\lambda, \lambda(T^\nu) = 1$. Let $\tilde{H}_2 = \mathcal{F}\overline{H}_2\mathcal{F}^{-1}$.

Proposition 2. *Operator $\tilde{H}_2$ is a bounded self-adjoint operator and it acts in the space $\tilde{\mathcal{H}}_2$ according to the formula*

$$(\tilde{H}_2 f)(x; y) = h(x; y)f(x; y) + \int_{T^\nu} h_1(x; y; t)f(t; x + y - t)dt, \tag{6}$$

where $h(x; y) = 16(J - 4J_1 + 16J_2 - 64J_3)\sum_{i=1}^{\nu}(1 - cos\frac{x_i+y_i}{2}cos\frac{x_i-y_i}{2})$,

$$h_1(x; y; t) = -24(J_1 - 11J_2 + 93J_3)\times$$

$$\times\sum_{i=1}^{\nu}[1 - 2cos\frac{x_i + y_i}{2}cos\frac{x_i - y_i}{2} + cos(x_i + y_i)] - 4(J + 5J_1 - 83J_2 + 773J_3)\times$$

$$\times\sum_{i=1}^{\nu}[cos\frac{x_i - y_i}{2} - cos\frac{x_i + y_i}{2}]cos(\frac{x_i + y_i}{2} - t_i), x, y, t \in T^\nu.$$

Proof. The proof is by direct calculation in which we use the Fourier transformation in formula (4). □

It follows from Lemma 1 and from this fact that to investigate spectrum of the operator H_2 in the space $\mathcal{H}_2$, it suffices to investigate the spectrum of the operator $\tilde{H}_2$ acting in the space $L_2((T^\nu)^2)$ according to formula (6).

The following fact is important for further investigating the spectrum of the operator $\tilde{H}_2$. Let the full quasi momentum of the system $x + y = \Lambda$ be fixed. Let $L_2(\Gamma_\Lambda)$ be the space of functions that are quadratically integrable over the manifold $\Gamma_\Lambda = \{(x; y) : x + y = \Lambda\}$. It is known [20] that the operators $\tilde{H}_2$ and the space $\tilde{\mathcal{H}}_2$ can be expanded into the direct integrals

$$\tilde{H}_2 = \int_{T^\nu} \bigoplus \tilde{H}_{2\Lambda} d\Lambda, \tilde{\mathcal{H}}_2 = \int_{T^\nu} \bigoplus \tilde{\mathcal{H}}_{2\Lambda} d\Lambda$$

of the operators $\tilde{H}_{2\Lambda}$ and the space $\tilde{\mathcal{H}}_{2\Lambda}$ such that the spaces $\tilde{\mathcal{H}}_{2\Lambda}$ are invariant with respect to the operators $\tilde{H}_{2\Lambda}$ and the operator $\tilde{H}_{2\Lambda}$ acts in the space $\tilde{\mathcal{H}}_{2\Lambda}$ according to the formula

$$(\tilde{H}_{2\Lambda} f_\Lambda)(x) = h_\Lambda(x)f_\Lambda(x) + \int_{T^\nu} h_{1\Lambda}(x; t)f_\Lambda(t)dt,$$

where $h_\Lambda(x)=h(x;\Lambda-x), h_{1\Lambda}(x;t)=h_1(x;\Lambda-x;t)$ and $f_\Lambda(x)=f(x;\Lambda-x)$.It is known that the continuous spectrum of the operator $\tilde{H}_{2\Lambda}$ consists of the intervals $G_\Lambda=[m_\Lambda;M_\Lambda]$, where $m_\Lambda=min_x h_\Lambda(x), M_\Lambda=max_x h_\Lambda(x)$.

The eigenfunction $\varphi_\Lambda\in L_2(T^\nu)$ of the operator $\tilde{H}_{2\Lambda}$ corresponding to the eigenvalue $z_\Lambda\notin G_\Lambda$ is called the BS the operator $\tilde{H}_2$, and the quantity z_Λ is called the energy of this BS.

We consider the operator K_Λ,

$$(K_\Lambda(z)f_\Lambda)(x)=\int_{T^\nu}\frac{h_{1\Lambda}(x;t)}{h_\Lambda(t)-z}f_\Lambda(t)dt.$$

This operator is totally continuous in the space $\mathcal{H}_{2\Lambda}$ for values of z not belonging to the set $G_\Lambda=Imh_\Lambda(x)=[m_\Lambda;M_\Lambda]$. Let $\Delta_\Lambda^\nu(z)=detD$, where

$$D=\begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3}.... & d_{1,\nu+1}\\ d_{2,1} & d_{2,2} & d_{2,3}.... & d_{2,\nu+1}\\ \vdots & \vdots & \vdots.... & \vdots\\ d_{\nu+1,1} & d_{\nu+1,2} & d_{\nu+1,3}.... & d_{\nu+1,\nu+1}\end{pmatrix}.$$

Here

$$d_{1,1}=1-24B\int_{T^\nu}\frac{g_\Lambda(s)ds_1ds_2...ds_\nu}{h_\Lambda(s)-z},$$

$$d_{k+1,1}=-24B\int_{T^\nu}\frac{f_{\Lambda_k}(s_k)g_\Lambda(s)ds_1ds_2...ds_\nu}{h_\Lambda(s)-z}, k=1,2,...,\nu,$$

$$d_{1,k+1}=-4C\int_{T^\nu}\frac{\varphi_{\Lambda_k}(s_k)ds_1ds_2...ds_\nu}{h_\Lambda(s)-z}, k=1,2,...,\nu,$$

$$d_{k+1,k+1}=1-4C\int_{T^\nu}\frac{f_{\Lambda_k}(s_k)\varphi_{\Lambda_k}(s_k)ds_1ds_2...ds_\nu}{h_\Lambda(s)-z}, k=1,2,...,\nu,$$

$$d_{k+1,i+1}=-4C\int_{T^\nu}\frac{f_{\Lambda_k}(s_k)\varphi_{\Lambda_i}(s_i)}{h_\Lambda(s)-z}ds_1ds_2...ds_\nu,$$
$$k=1,2,...,\nu, i=1,2,...,\nu, i\neq k.$$

In these formulas,

$$g_\Lambda(s)=\sum_{i=1}^{\nu}[1+\cos\Lambda_i-2\cos\frac{\Lambda_i}{2}\cos(\frac{\Lambda_i}{2}-S_i)],$$

$$\varphi_{\Lambda_k}(s_k)=cos(\frac{\Lambda_k}{2}-s_k)-cos\frac{\Lambda_k}{2},$$

$$f_{\Lambda_k}(s_k) = cos(\frac{\Lambda_k}{2} - s_k), k = 1, 2, ..., \nu, \Lambda \in T^\nu, s \in T^\nu.$$

Lemma 2. *A number $z = z_0 \notin G_\Lambda$ is an eigenvalue of the operator $\tilde{H}_{2\Lambda}$ if and only if it is a zero of the function $\Delta^\nu_\Lambda(z)$, i. e., $\Delta^\nu_\Lambda(z_0) = 0$.*

Proof. In the case under consideration, the equation for the eigenvalues is an integral equation with a degenerate kernel. It is therefore equivalent to a system of linear homogeneous algebraic equations. It is known that such a system has a nontrivial solution if and only if its determinant is equal to zero. In this case, the determinant of this linear homogeneous algebraic system is equal to function $\Delta^\nu_\Lambda(z)$. □

We set $A = J - 4J_1 + 16J_2 - 64J_3, B = J_1 - 11J_2 + 93J_3, C = J + 5J_1 - 83J_2 + 773J_3$.

Theorem 1. *Let $A = 0$ and ν be arbitrary. Then the operator $\tilde{H}_2$ has two BSs φ_1 and φ_2 (not taking the order of the energy degeneration into account) with the energy values $z_1 = -18B$ and $z_2 = -24B[\nu + 1 + \sum_{i=1}^{\nu} cos\Lambda_i]$ and z_1 is degenerate $\nu - 1$ times, while z_2 is not degenerate, $z_i \notin G_\Lambda, i = 1; 2$, for all $\Lambda \in T^\nu$, i.e., the energy values of these BSs lie outside the continuous spectrum domain of the operator $\tilde{H}_{2\Lambda}$. When $B = 0$, this BSs vanishes because it is incorporated into the continuous spectrum.*

Proof. If $A = 0$, then $h_\Lambda(s) \equiv 0$, and

$$\Delta^\nu_\Lambda(z) = (1 + \frac{2C}{z})^{\nu-1} \times \{(1 + \frac{2C}{z})[1 + \frac{24B}{z}\sum_{i=1}^{\nu}(1 + cos\Lambda_i)] -$$

$$-\frac{96BC}{z^2}\sum_{i=1}^{\nu} cos^2\frac{\Lambda_i}{2}\}.$$

and $C = 9B$. Solving the equation $\Delta^\nu_\Lambda(z) = 0$, we prove the theorem. □

Note. *In the theorem, the zeroth-order degeneration corresponds to the case where there is no BS.*

Let $\pi = (\pi; \pi; ...; \pi) \in T^\nu$.

Theorem 2. *Let $\Lambda = \pi, C \neq 0$. Then the operator $\tilde{H}_2$ has only one BS φ with the energy value $z = 16\nu A - 2C$ and this energy level is degenerate ν times. In addition, if $C > 0$, then $z < m_\Lambda$, and if $C < 0$, then $z > M_\Lambda$. When $C = 0$, this BS vanishes because it is incorporated into the continuous spectrum.*

Proof. The proof is based on the equality $h_\Lambda(x) = 16\nu A$ with $\Lambda = \pi$ and also on the corresponding form of the function $\Delta^\nu_\Lambda(z)$. $\square$

Theorem 3. *Let $C = 0$ and ν be an arbitrary number. Then the operator $\tilde{H}_2$ has at most one BS, and the corresponding energy level is not degenerated.*

Proof. If $C = 0$, the relations $A = -9B$, and $h_{1\Lambda}(x;t) = -24B\sum_{i=1}^{\nu}[1-2cos\frac{\Lambda_i}{2}cos(\frac{\Lambda_i}{2} - x_i) + cos\Lambda_i], h_\Lambda(x) = 16A\sum_{i=1}^{\nu}[1 - cos\frac{\Lambda_i}{2}cos(\frac{\Lambda_i}{2} - x_i)]$ hold. Using the form determinant $\Delta^\nu_\Lambda(z)$ and solving the corresponding equation, we obtain the statement in Theorem 3. $\square$

Denote the four $(J; J_1; J_2; J_3)$ by P and introduce the following ranges of the four P for $\nu = 1$;
$F_1 = \{P : A < 0, B < 0, C < 0\}, F_2 = \{P : A > 0, B > 0, C > 0\}, F_3 = \{P : A > 0, B > 0, C < 0\}$, $F_4 = \{P : A < 0, B < 0, C > 0\}, F_5 = \{P : A < 0, B > 0, C < 0\}, F_6 = \{P : A > 0, B < 0, C > 0\}$, $F_7 = \{P : B = 0, A = C > 0\}, F_8 = \{P : B = 0, A = C < 0\}$.

In the case where $\nu = 1$, the change of the energy spectrum is described by the following theorems.

Theorem 4.

1. Let $P \in F_1$ and $\Lambda \in]0; \pi[\quad (\Lambda \in]\pi; 2\pi[)$

a) If $C \neq 12B$ then the operator $\tilde{H}_2$ has two BSs φ_1 and φ_2 with the corresponding energy levels $z_1 < m_\Lambda$ and $z_2 > M_\Lambda$.

b) If $C = 12B$ then the operator $\tilde{H}_2$ has only one BS φ with the energy level $z < m_\Lambda$.

2. Let $P \in F_2$ and $\Lambda \in]0; \pi[\quad (\Lambda \in]\pi; 2\pi[)$

a) If $4A < C < 12B, cos\frac{\Lambda}{2} > \frac{C}{12B}, \quad (4A < 12B < C)$, then the operator $\tilde{H}_2$ has three BSs $\varphi_i, i = 1, 2, 3;$ with the corresponding energy values $z_i < m_\Lambda, i = 1, 2; z_3 > M_\Lambda$.

b) If $C < 4A < 12B, cos\frac{\Lambda}{2} > \frac{C}{12B}, \quad (C > 12B = 4A)$, then the operator $\tilde{H}_2$ has two BSs $\varphi_i, i = 1, 2;$ with the corresponding energy values $z_1 < m_\Lambda, z_2 > M_\Lambda$. In this case third BS vanishes because it is incorporated into the continuous spectrum.

c) If $C < 12B < 4A, cos\frac{\Lambda}{2} > \frac{C}{12B}, \quad (C > 12B, A > 3B)$ then the operator $\tilde{H}_2$ has only one BS φ with energy value $z > M_\Lambda$.

d) If $C = 12B$ then the operator $\tilde{H}_2$ has only one BS φ with energy value $z < m_\Lambda$.

e) If $C > 12B \quad (C < 12B)$ then the operator $\tilde{H}_2$ has two BSs φ_1, φ_2 with corresponding energy values $z_1 < m_\Lambda, z_2 > M_\Lambda$.

3. Let $P \in F_3$ *and* $\Lambda \in]0;\pi[\quad (\Lambda \in]\pi;2\pi[)$.

a) If $C \geq -12B$ *then the operator* $\tilde{H}_2$ *has a two BSs* φ_1 *and* φ_2 *with the corresponding energy values* $z_1 < m_\Lambda$ *and* $z_2 > M_\Lambda$.

b) If $C < -12B$ *then the operator* $\tilde{H}_2$ *has only one BS* φ *with energy value* $z < m_\Lambda$.

4. Let $P \in F_4$ *and* $\Lambda \in]0;\pi[\quad (\Lambda \in]\pi;2\pi[)$

a) If $4A - 12B - C > 0, cos\frac{\Lambda}{2} > \frac{C}{4A-12B-C} \quad (cos\frac{\Lambda}{2} \neq \frac{C}{12B})$, *then the operator* $\tilde{H}_2$ *has three (two) BSs* $\varphi_i, i = 1,2,3; (\varphi_j, j = 1,2;)$ *with the corresponding energy values* $z_k < m_\Lambda, k = 1,2; z_3 > M_\Lambda \quad (z_1 < m_\Lambda, z_2 > M_\Lambda)$.

b) If $4A - 12B - C > 0, -\frac{C}{12B} < cos\frac{\Lambda}{2} < \frac{C}{4A-12B-C}$ *or* $4A - 12B - C < 0 \quad (cos\frac{\Lambda}{2} = \frac{C}{12B})$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z > M_\Lambda$.

5. Let $P \in F_5$ *and* $\Lambda \in]0;\pi[\quad (\Lambda \in]\pi;2\pi[)$.

a) If $cos\frac{\Lambda}{2} > -\frac{C}{12B}, C \geq 4A \quad (cos\frac{\Lambda}{2} < \frac{C}{12B}, C \geq 4A)$, *then the operator* $\tilde{H}_2$ *has three BSs* $\varphi_1, \varphi_2, \varphi_3$ *with corresponding energy levels* $z_i < m_\Lambda, i = 1,2; z_3 > M_\Lambda$.

b) If $C < 4A, 4A - 12B - C < 0, cos\frac{\Lambda}{2} > \frac{C}{4A-12B-C} \quad (C < 4A, 4A - 12B - C < 0, cos\frac{\Lambda}{2} < -\frac{C}{4A-12B-C})$, *then the operator* $\tilde{H}_2$ *has three BSs* $\varphi_1, \varphi_2, \varphi_3$ *with the corresponding energy values* $z_i < m_\Lambda, i = 1,2; z_3 > M_\Lambda$.

c) If $C < 4A, 4A - 12B - C < 0, -\frac{C}{12B} < cos\frac{\Lambda}{2} \leq \frac{C}{12B} \quad (C < 4A, 4A - 12B - C < 0, -\frac{C}{4A-12B-C} \leq cos\frac{\Lambda}{2} < \frac{C}{12B})$, *or* $C < 4A, 4A - 12B - C \geq 0, \quad (C > 4A, 4A - 12B - C \geq 0)$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with energy value* $z > M_\Lambda$.

d) If $cos\frac{\Lambda}{2} = -\frac{C}{12B}, C \geq 4A \quad (cos\frac{\Lambda}{2} = \frac{C}{12B}, C \geq 4A)$, *then the operator* $\tilde{H}_2$ *has two BSs* $\varphi_i, i = 1,2;$ *with the energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

e) If $cos\frac{\Lambda}{2} = -\frac{C}{12B}, C < 4A \quad (cos\frac{\Lambda}{2} > \frac{C}{12B}, C < 4A)$, *then operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z > M_\Lambda$.

f) If $cos\frac{\Lambda}{2} < -\frac{C}{12B} \quad (cos\frac{\Lambda}{2} > \frac{C}{12B})$, *then the operator* $\tilde{H}_2$ *has two BSs* $\varphi_i, i = 1,2;$ *with the corresponding energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

6. Let $P \in F_6$ *and* $\Lambda \in]0;\pi[\quad (\Lambda \in]\pi;2\pi[)$.

a) If $cos\frac{\Lambda}{2} < -\frac{C}{12B} \quad (cos\frac{\Lambda}{2} > \frac{C}{12B})$, *then the operator* $\tilde{H}_2$ *has two BSs* φ_1, φ_2 *with the corresponding energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

b) If $cos\frac{\Lambda}{2} \geq -\frac{C}{12B} \quad (cos\frac{\Lambda}{2} \leq \frac{C}{12B})$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z < m_\Lambda$.

7. Let $P \in F_7$ *and* $\Lambda \in]0;\pi[\quad (\Lambda \in]\pi;2\pi[)$.

Then the operator $\tilde{H}_2$ *has two BSs* φ_1, φ_2 *with the corresponding energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

8. Let $P \in F_8$ *and* $\Lambda \neq 0$.

Then the operator $\tilde{H}_2$ *has two BSs* φ_1, φ_2 *with the corresponding energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

Theorem 5. *Let* $\Lambda = 0$.

1. a) If $P \in F_1, C > 12B$ *then the operator* $\tilde{H}_2$ *has two BSs* φ_1, φ_2 *with the corresponding energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

b) If $P \in F_1, C \leq 12B$ *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z < m_\Lambda$.

2. a)If $P \in F_2, 4A < C < 12B$, *then the operator* $\tilde{H}_2$ *has three BSs* $\varphi_i, i = 1,2,3;$ *with the corresponding energy values* $z_j < m_\Lambda, j = 1,2; z_3 > M_\Lambda$.

b) If $P \in F_2, C \leq 4A, C < 12B$, *or* $P \in F_2, 4A < 12B < C$, *then the operator* $\tilde{H}_2$ *has two BSs* $\varphi_i, i = 1,2;$ *with the corresponding energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

c) If $P \in F_2, C = 12B, C > 4A$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z < m_\Lambda$.

d) If $P \in F_2, C = 4A \geq 12B$, *or* $P \in F_2, 12B < 4A < C$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z > M_\Lambda$.

e) If $P \in F_2, C = 12B < 4A$, *or* $P \in F_2, 12B < 4A < C$, *then the operator* $\tilde{H}_2$ *has no BS.*

3. a) If $P \in F_3, C < -12B, A \geq 3B$, *then the operator* $\tilde{H}_2$ *has two BSs* $\varphi_i, i = 1,2;$ *with the corresponding energy values* $z_1 < m_\Lambda, z_2 > M_\Lambda$.

b) If $P \in F_3, A < 3B$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z > M_\Lambda$.

c) If $P \in F_3, C \geq -12B, A \geq 3B$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z < m_\Lambda$.

4. a) If $P \in F_4, C > -12B$, *then the operator* $\tilde{H}_2$ *has two BSs* $\varphi_i, i = 1,2;$ *with the corresponding energy values* $z_i < m_\Lambda, i = 1,2;$

b) If $P \in F_4, C = -12B$, *then the operator* $\tilde{H}_2$ *has no BS.*

c) If $P \in F_4, C < -12B$, *then the operator* $\tilde{H}_2$ *has only one BS* φ *with the energy value* $z < m_\Lambda$.

5. a) If $P \in F_5, -12B < C < 4A, C > 2A - 6B$, *then the operator* $\tilde{H}_2$ *has two BSs* $\varphi_i, i = 1,2;$ *with the corresponding energy values* $z_i < m_\Lambda, i = 1,2;$

b) If $P \in F_5, -12B < C < 4A, C \leq 2A - 6B$, or $P \in F_5, C = -12B < 4A$, then the operator $\tilde{H}_2$ has no BS.

c) If $P \in F_5, C = -12B \geq 4A$, or $P \in F_5, C < -12B$, then the operator $\tilde{H}_2$ has only one BS with the energy value $z < m_\Lambda$.

6. a) If $P \in F_6, 4A \leq C < -12B$, then the operator $\tilde{H}_2$ has two BSs $\varphi_i, i = 1, 2$; with the corresponding energy values $z_i > M_\Lambda, i = 1, 2$.

b) If $P \in F_6, C = 4A > -12B$, or $P \in F_6, C = -12B \geq 4A$, or $P \in F_6, C < -12B, C < 4A$, then the operator $\tilde{H}_2$ has no BS.

c) If $P \in F_6, C = -12B < 4A$, or $P \in F_6, C > -12B, C \neq 4A$, then the operator $\tilde{H}_2$ has only one BS φ with the energy value $z > M_\Lambda$.

7. If $P \in F_7$ $(P \in F_8)$, then the operator $\tilde{H}_2$ has only one BS φ with the energy value $z > M_\Lambda (z < m_\Lambda)$.

A sketch proof of Theorems 4-5 is given below. In the case under consideration, the equation for the eigenvalues is an integral equation with a degenerate kernel. It is therefore equivalent to a system of linear homogeneous algebraic equations. It is known that such a system has a nontrivial solution if and only if its determinant is equal to zero. In this case, the equation $\Delta^\nu_\Lambda(z) = 0$ is therefore equivalent to the equation stating that the determinant of the system is zero. Expressing all integrals in the equation $\Delta^\nu_\Lambda(z) = 0$ through the integral

$$J(z) = \int_T \frac{dt}{h_\Lambda(t) - z},$$

we find that the equation $\Delta^\nu_\Lambda(z) = 0$ is equivalent to the equation

$$J(z) = \{-C(z - 16A) + 16A(4A - 12B - C)cos^2\frac{\Lambda}{2}\}\times \tag{7}$$

$$\times\{C(z - 16A)^2 + 16A(12B + C)(z - 16A)cos^2\frac{\Lambda}{2} + 3072A^2Bcos^4\frac{\Lambda}{2}\}^{-1}.$$

Because $\frac{1}{h_\Lambda(t)-z}$ is a continuous function for $z \notin G_\Lambda$ and

$$[J(z)]' = \int_T \frac{dt}{[h_\Lambda(t) - z]^2} > 0,$$

the function $J(z)$ is an increasing function of z for $z \notin G_\Lambda$. Moreover, $J(z) \to 0$ as $z \to -\infty$, $J(z) \to +\infty$ as $z \to m_\Lambda - 0$, $J(z) \to -\infty$ as $z \to M_\Lambda + 0$, and $J(z) \to 0$ as $z \to +\infty$. Analysis of Eq. (7) outside the set $G_\Lambda = [m_\Lambda; M_\Lambda]$ leads to the proof of Theorems 4-5.

The energy spectrum in the case where $\nu=2$ for the full quasi momenta of the form $\Lambda=(\Lambda_1;\Lambda_2)=(\Lambda_0;\Lambda_0)$ is described below. It is easy to see that if the parameters J, J_1, J_2, J_3 and Λ_0 satisfy the conditions of Theorems 4-5, the statements of the theorems are true. Only one additional BS Ψ appears, whose energy value is $\tilde{z}<m_\Lambda$ or $\tilde{z}_\Lambda>M_\Lambda$ if $C>0$ or $C<0$. If $C=0$, the operator $\tilde{H}_2$ has no additional BS.

The proof of this statement is based on the fact that if $\nu=2$ and $\Lambda=(\Lambda_0;\Lambda_0)$, then the function $\Delta_\Lambda^\nu(z)$ has the form

$$\Delta_\Lambda^\nu(z)=[1-4C\int_{T^2}\frac{[\cos(\frac{\Lambda_0}{2}-s_1)-\cos(\frac{\Lambda_0}{2}-s_2)]^2}{h_\Lambda(s)-z}ds]\times\Psi(z), \tag{8}$$

where

$$\Psi(z)=\{1-48B\int_{T^2}\frac{1+cos\Lambda_0-cos\frac{\Lambda_0}{2}[\cos(\frac{\Lambda_0}{2}-s_1)+\cos(\frac{\Lambda_0}{2}-s_2)]}{h_\Lambda(s_1;s_2)-z}ds_1ds_2\}\times \tag{9}$$

$$\times\{1-8C\int_{T^2}\frac{cos(\frac{\Lambda_0}{2}-s_1)[\cos(\frac{\Lambda_0}{2}-s_1)+cos(\frac{\Lambda_0}{2}-s_2)-2cos\frac{\Lambda_0}{2}]}{h_\Lambda(s_1;s_2)-z}ds_1ds_2\}-$$

$$-768BC\int_{T^2}\frac{cos(\frac{\Lambda_0}{2}-s_1)-cos\frac{\Lambda_0}{2}}{h_\Lambda(s_1;s_2)-z}ds_1ds_2\times$$

$$\times\int_{T^2}\frac{\cos(\frac{\Lambda_0}{2}-s_1)\times\{1+\cos\Lambda_0-\cos\frac{\Lambda_0}{2}[cos(\frac{\Lambda_0}{2}-s_1)+cos(\frac{\Lambda_0}{2}-s_2)]\}}{h_\Lambda(s_1;s_2)-z}ds_1ds_2.$$

The equation $\Delta_\Lambda^\nu(z)=0$ is therefore equivalent to the equation

$$1-4C\int_{T^2}\frac{[\cos(\frac{\Lambda_0}{2}-s_1)-\cos(\frac{\Lambda_0}{2}-s_2)]^2}{h_\Lambda(s_1;s_2)-z}ds_1ds_2=0 \tag{10}$$

and

$$\psi_\lambda(z)=0. \tag{11}$$

It is easy to see Eq.(10) has a unique solution $\tilde{z}<m_\Lambda$ if $C>0$; if $C<0$, this solution satisfies the condition $\tilde{z}>M_\Lambda$. If $C=0$, Eq. (10) has no solution. Expressing the integrals in Eq. (11) through the integral

$$J(z)=\int_{T^2}\frac{ds_1ds_2}{h_\Lambda(s_1;s_2)-z},$$

we obtain an equation of the form

$$\eta_\Lambda(z)J(z) = \xi_\Lambda(z),$$

where

$$\eta_\Lambda(z) = C(z-32A)^2 + 32A(C+6B)cos^2\frac{\Lambda_0}{2}(z-32A) + 6144A^2Bcos^4\frac{\Lambda_0}{2}$$

and

$$\xi_\Lambda(z) = -C(z-32A) + 32A(3A-6B-C)cos^2\frac{\Lambda_0}{2}.$$

In turn, for $\eta_\Lambda(z) \neq 0$, the latter equation is equivalent to the equation of the form

$$J(z) = \frac{\xi_\Lambda(z)}{\eta_\Lambda(z)}. \tag{12}$$

Analyzing Eq. (12) outside the set G_Λ and taking into account that the function $J(z)$ is monotonic for $z \notin [m_\Lambda; M_\Lambda]$, we obtain statements similar to the statements in Theorems 4-5.

For all other quasi momenta $\Lambda = (\Lambda_1;\Lambda_2), \Lambda_1 \neq \Lambda_2$, these exists sets $G_j, j = \overline{0,5}$, of the parameters J, J_1, J_2, J_3 and Λ such that in every set G_j the operator $\tilde{H}_2$ has exactly j BSs (taking the energy degeneration order into account) with the corresponding energy values $z_k \notin G_\Lambda, k = \overline{1,5}$.

Indeed, in this case and for $\nu = 2$, the function $\Delta^\nu_\Lambda(z)$ has the form $\Delta^\nu_\Lambda(z) = detD$, where

$$D = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{pmatrix},$$

here

$$d_{1,1} = 1 - 24B\int_{T^2}\frac{2-2\sum_{i=1}^2 cos\frac{\Lambda_i}{2}cos(\frac{\Lambda_i}{2}-s_i) + \sum_{i=1}^2 cos\Lambda_i}{h_\Lambda(s)-z}ds_1ds_2,$$

$$d_{1,k+1} = -4C\int_{T^2}\frac{cos(\frac{\Lambda_k}{2}-s_k) - cos\frac{\Lambda_k}{2}}{h_\Lambda(s)-z}ds_1ds_2, k = 1,2;$$

$$d_{k+1,1} = -24B\times$$

$$\times\int_{T^2}\frac{cos(\frac{\Lambda_k}{2}-s_k)[2-2\sum_{i=1}^{2}cos\frac{\Lambda_i}{2}cos(\frac{\Lambda_i}{2}-s_i)+\sum_{i=1}^{2}cos\Lambda_i}{h_\Lambda(s)-z}ds_1ds_2,$$

$$k=1,2;$$

$$d_{k+1,k+1}=1-4C\int_{T^2}\frac{cos^2(\frac{\Lambda_k}{2}-s_k)-cos\frac{\Lambda_k}{2}cos(\frac{\Lambda_k}{2}-s_k)}{h_\Lambda(s)-z}ds_1ds_2, k=1,2;$$

$$d_{2,3}=-4C\int_{T^2}\frac{cos(\frac{\Lambda_1}{2}-s_1)cos(\frac{\Lambda_2}{2}-s_2)-cos\frac{\Lambda_2}{2}cos(\frac{\Lambda_1}{2}-s_1)}{h_\Lambda(s)-z}ds_1ds_2,$$

$$d_{3,2}=-4C\int_{T^2}\frac{cos(\frac{\Lambda_1}{2}-s_1)cos(\frac{\Lambda_2}{2}-s_2)-cos\frac{\Lambda_1}{2}cos(\frac{\Lambda_2}{2}-s_2)}{h_\Lambda(s)-z}ds_1ds_2,$$

$$\Lambda\in T^2, s\in T^2.$$

Expressing all integrals in the equation $\Delta_\Lambda^\nu(z)=0$ through $J(z)$ and rearranging algebraically, we reduce the latter equation to the form

$$\Theta_\Lambda(z)J(z)=\chi_\Lambda(z), \tag{13}$$

where $\Theta_\Lambda(z)$ is the fifth-order polynomial in z and $\chi_\Lambda(z)$ is a lower-order polynomial in z. Analyzing Eq. (13) outside the set G_Λ and taking into account that the function $J(z)$ with $z\notin[m_\Lambda;M_\Lambda]$ is monotonic, we can easily verify that the equation has no more that five solutions outside the set G_Λ.

We now consider the case of $\nu=3$. Let the full quasi momentum have the form $\Lambda=(\Lambda_1,\Lambda_2,\Lambda_3)=(\Lambda_0,\Lambda_0,\Lambda_0)$. If the parameters Λ_0, J, J_1, and J_2 satisfy the conditions in Theorems 4-5, then statements similar to those in the theorems are true. Only one additional BS η appears, whose energy value is $\tilde{z}$. This energy level is twice degenerate and $\tilde{z}<m_\Lambda$ or $\tilde{z}>M_\Lambda$ if $C>0$ or $C<0$. This additional BS vanishes when $C=0$ because it is incorporated into the continuous spectrum.

To prove this, we note that in this case, the function $\Delta_\Lambda^\nu(z)$ has the form

$$\Delta_\Lambda^\nu(z)=[1-4C\int_{T^3}\frac{[cos(\frac{\Lambda_0}{2}-s_1)-cos(\frac{\Lambda_0}{2}-s_2)]^2}{h_\Lambda(s)-z}ds_1ds_2ds_3]^2\tilde{\psi}_\Lambda(z), s\in T^3,$$

where

$$\tilde{\psi}_\lambda(z) = [1-24B\int_{T^3}\frac{[3+3cos\Lambda_0-2cos\frac{\Lambda_0}{2}(\sum_{i=1}^3 cos(\frac{\Lambda_0}{2}-s_i))]}{h_\Lambda(s)-z}ds_1ds_2ds_3]\times$$

$$\times\{1-4C\int_{T^3}\frac{cos(\frac{\Lambda_0}{2}-s_1)[\sum_{i=1}^3 cos(\frac{\Lambda_0}{2}-s_i)-3cos\frac{\Lambda_0}{2}]}{h_\Lambda(s)-z}ds_1ds_2ds_3\}-$$

$$-288BC\times$$

$$\times\int_{T^3}\frac{cos(\frac{\Lambda_0}{2}-s_1)[3+3cos\Lambda_0-2cos\frac{\Lambda_0}{2}\sum_{i=1}^3 cos(\frac{\Lambda_0}{2}-s_i)]}{h_\Lambda(s)-z}ds_1ds_2ds_3\times$$

$$\times\int_{T^3}\frac{cos(\frac{\Lambda_0}{2}-s_1)-cos\frac{\Lambda_0}{2}}{h_\Lambda(s)-z}ds_1ds_2ds_3.$$

Therefore the equation $\Delta_\Lambda^\nu(z)=0$ is equivalent to the equations

$$[1-4C\int_{T^3}\frac{[\cos(\frac{\Lambda_0}{2}-s_1)-\cos(\frac{\Lambda_0}{2}-s_2)]^2ds_1ds_2ds_3}{h_\Lambda(s)-z}]^2=0 \qquad (14)$$

and

$$\tilde{\psi}_\Lambda(z)=0. \qquad (15)$$

It is easy to see that Eq. (14) has a unique double solution $z^{'}$ if $C\neq 0$ and $z^{'}<m_\Lambda$ or $z^{'}>M_\Lambda$ if $C>0$ or $C<0$. Expressing all integrals in Eq. (15) through $J(z)=\int_{T^3}\frac{ds_1ds_2ds_3}{h_\Lambda(s)-z}$, we obtain the equation

$$\tilde{\eta}_\Lambda(z)J(z)=\tilde{\Theta}_\Lambda(z), \qquad (16)$$

where

$$\tilde{\eta_\Lambda}(z)=C(z-48A)^2+48A(C+6B)cos^2\frac{\Lambda_0}{2}(z-48A)+13824A^2Bcos\frac{\Lambda_0}{2}$$

and

$$\tilde{\Theta_\Lambda}(z)=-C(z-48A)+48A(3A-C-6B)cos^2\frac{\Lambda_0}{2}.$$

If $\tilde{\eta_\Lambda}(z)\neq 0$, Eq.(16) is, in turn, equivalent to the equation

$$J(z)=\frac{\tilde{\Theta_\Lambda}(z)}{\tilde{\eta_\Lambda}(z)}. \qquad (17)$$

Analyzing Eq. (17) outside the set G_Λ and taking into account that the function $J(z)$ for $z \notin G_\Lambda$ is monotonic, we prove the statements made above.

If $\Lambda \neq (\Lambda_0, \Lambda_0, \Lambda_0)$, the system has at most seven BSs (taking the energy degeneration order into account), and there exist sets $G_k, k = \overline{0,7}$, of the parameters Λ, J, J_1, J_2 and J_3 such that in every set G_k, the system has exactly k BSs. The energy values of these BSs lie outside the set G_Λ. When passing from one of these sets to another, either some additional BSs of the operator $\tilde{H}_2$ appear or some existing BSs vanish. In this case, function $\Delta_\Lambda^\nu(z)$ has the form $\Delta_\Lambda^\nu(z) = detD$, where

$$D = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} \end{pmatrix}.$$

Here

$$d_{1,1} = 1 - 24B \int_{T^3} \frac{g_\Lambda(s)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3,$$

$$d_{1,k+1} = -4C \int_{T^3} \frac{f_{\Lambda_k}(s_k)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3, k = 1, 2, 3;$$

$$d_{k+1,1} = -24B \int_{T^3} \frac{\varphi_{\Lambda_k}(s_k) g_\Lambda(s)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3, k = 1, 2, 3;$$

$$d_{k+1,k+1} = 1 - 4C \int_{T^3} \frac{f_{\Lambda_k}(s_k)\varphi_{\Lambda_k}(s_k)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3, k = 1, 2, 3;$$

$$d_{2,2+k} = -4C \int_{T^3} \frac{\varphi_{\Lambda_1}(s_1) f_{\Lambda_k}(s_k)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3, k = 1, 2;$$

$$d_{k+2,2} = -4C \int_{T^3} \frac{\varphi_{\Lambda_k}(s_k) f_{\Lambda_1}(s_1)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3, k = 1, 2;$$

$$d_{3,4} = -4C \int_{T^3} \frac{\varphi_{\Lambda_2}(s_2) f_{\Lambda_3}(s_3)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3,$$

$$d_{4,3} = -4C \int_{T^3} \frac{\varphi_{\Lambda_3}(s_3) f_{\Lambda_2}(s_2)}{h_\Lambda(s) - z} ds_1 ds_2 ds_3.$$

In these formulas,

$$g_\Lambda(s) = \sum_{i=1}^{3} [1 + cos\Lambda_i - 2cos\frac{\Lambda_i}{2} cos(\frac{\Lambda_i}{2} - s_i)],$$

$$f_{\Lambda_k}(s_k) = cos(\frac{\Lambda_k}{2} - s_k) - cos\frac{\Lambda_k}{2}, \quad \varphi_{\Lambda_k}(s_k) = cos(\frac{\Lambda_k}{2} - s_k), k = 1, 2, 3.$$

Expressing all integrals in the equation $\Delta_\Lambda^\nu(z) = 0$ through $J(z)$ and rearranging algebraically, we reduce this equation to the form $J(z) = \frac{N_\Lambda(z)}{M_\Lambda(z)}$, where $M_\Lambda(z)$ is a seventh-order polynomial in z, and $N_\Lambda(z)$ is a lower-order polynomial in z. Therefore, this equation has no more than seven solutions outside the set G_Λ.

For an arbitrary $\nu > 3$ and $\Lambda = (\Lambda_1; \Lambda_2; ...; \Lambda_\nu) = (\Lambda_0; \Lambda_0; ...; \Lambda_0)$ the change of energy operator spectrum is similar to that observed in the case of $\nu = 1$. In this case the operator $\tilde{H}_2$ with $C \neq 0$ has only one additional BS. The energy z of this additional BS is degenerated $\nu - 1$ times. For all other values of the full quasi momentum Λ of the system, the operator $\tilde{H}_2$ has no more that $2\nu+1$ BSs (taking the energy degeneracy order into account) with energy values lying outside the set G_Λ.

The proof of these statements is based on finding zeros of the function $\Delta_\Lambda^\nu(z)$. Expressing all integrals in $\Delta_\Lambda^\nu(z)$ through $J(z)$, we can bring the equation $\Delta_\Lambda^\nu(z) = 0$ to the form

$$J(z) = \frac{C_\Lambda(z)}{D_\Lambda(z)}, \tag{18}$$

where $D_\Lambda(z)$ is a $(2\nu + 1)$ th-order polynomial in z and $C_\Lambda(z)$ is also a polynomial in z whose order (with respect to $D_\Lambda(z)$) is lower. Analysis of Eq. (18) outside the set G_Λ leads to the proof of the statements made above.

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ДОСЛІДЖЕННЯ СПЕКТРА І ЗВ'ЯЗАНИХ СТАНІВ ОПЕРАТОРА ЕНЕРГІЇ ДВОМАГНОННИХ СИСТЕМ У $\nu-$ВИМІРНОМУ НЕГЕЙЗЕНБЕРГІВСЬКОМУ ФЕРРОМАГНЕТИКУ ІЗ ЗНАЧЕННЯМ СПІНУ $S = 2$ ТА ІЗ ВЗАЄМОДІЄЮ НАЙБЛИЖЧИХ СУСІДІВ

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Розглядається двомагнонна система у $\nu-$вимірному негейзенбергівському ферромагнетику із значенням спіну $s = 2$ та із взаємодією найближчих сусідів. Досліджується спектр і зв'язаний стан (ЗС) для всіх значень повного квазіімпульса Λ, для довільних значень розмірності ґратки ν і для всіх значень параметрів гамільтоніана. Ми показуємо, що:

(i) для довільного $\nu \geq 2$ і для повного квазіімпульса у вигляді $\Lambda = (\Lambda_1; \Lambda_2; ...; \Lambda_\nu) = (\Lambda_0; \Lambda_0; ...; \Lambda_0)$ зміна енергетичного спектра системи досліджується аналогічно як у випадку $\nu = 1$. В цьому випадку оператор $\widetilde{H}_2$ має єдиний додатковий ЗС;

(ii) енергії цього додаткового ЗС z є $\nu - 1-$кратно виродженими;

(iii) якщо $\Lambda \neq (\Lambda_0; \Lambda_0; ...; \Lambda_0)$, то в $\nu-$вимірній ґратці система має не більше, ніж $2\nu + 1$ ЗС.