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THE CLASSICAL ABRAHAM-LORENTZ ELECTRON MASS THEORY LEGACY. PART I: THE AMPER LAW ASPECTS REVISITING

ANATOLIJ PRYKARPATSKI

The Department of Applied Mathematics at AGH University of Science and Technology of Krakow, Poland

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In Part 1 of the work new classical electrodynamics models of interacting charged point particles and related physical aspects are reviewed. Based on the geometry-free approach of vacuum field theory, developed by author, the Lagrangian and Hamiltonian reformulations of some alternative classical electrodynamics models are elaborated.

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У першій частині праці подано огляд нових класичних моделей взаємодіючих заряджених точкових частинок та деякі повязані з ними фізичні аспекти. Грунтуючись на розробленому автором негеометричному вакуумно-польовому підході, запропоновані як Лагранжеве так і Гамільтонове переформулювання альтернативних електродинамічних моделей.

1. Classical relativistic electrodynamics models revisiting: Lagrangian and Hamiltonian analysis

1.1. Introductory setting

It is nowadays considered [30, 41, 42, 45, 50] that classical electrodynamics is the most fundamental physical theory largely owing to the depth of its theoretical foun-

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E-mail: pryk.anat@cybergal.com, pryk.anat@ua.fm

dations and wealth of experimental verifications. In the review we describe a new aspects of the classical Maxwell theory, based on a vacuum field medium model, and reanalyze some of the modern classical electrodynamics problems related with the description of a charged point particle dynamics under external electromagnetic field. We remark here that under "*a charged point particle*" we as usually understand an elementary material charged particle whose internal spatial structure is assumed to be unimportant and is not taken into account, if the contrary is not specified. The Maxwell's equations serve as foundational to the whole modern electromagnetic theory. They are the cornerstone of a myriad of technologies and are basic to the understanding of innumerable effects. Yet there are a few effects or physical phenomena that cannot be explained [7, 8, 46, 55, 56, 62, 63, 82] within the conventional Maxwell theory. It is important to note here that in [43, 46, 44, 76] there is argued that the Maxwell equations as themselves do not determine causal related to each other electric and magnetic fields, which prove, in reality, to be generated independently by an external charge and current distributions: "There is a widespread interpretation of Maxwell's equations indicating that spatially varying electric and magnetic fields can cause each other to change in time, thus giving rise to a propagating electromagnetic wave..." However, Jefimenko's equations show an alternative point of view [45]. Jefimenko writes: "...neither Maxwell's equations nor their solutions indicate an existence of causal links between electric and magnetic fields. Therefore, we must conclude that an electromagnetic field is a *dual entity* always having an electric and a magnetic component simultaneously *created by their common sources*: time-variable electric charges and currents." Essential features of these equations are easily observed which are that the right hand sides involve "retarded" time which reflects the "causality" of the expressions. In other words, the left side of each equation is actually "caused" by the right side, unlike the normal differential expressions for Maxwell's equations, where both sides take place simultaneously. In the typical expressions for Maxwell's equations there is no doubt that both sides are equal to each other, but as Jefimenko notes [45], "... since each of these equations connects quantities simultaneous in time, none of these equations can represent a causal relation." The second feature is that the expression for (electric field) E does not depend upon (magnetic field) B and vice versa. Hence, it is impossible for E and B fields to be "creating" each other. Charge density and current density are creating them both. As the Efimenko's equations for the electric field E and the magnetic field B directly follow from the classical retarded Lienard–Wiechert potentials, generated by physically real external charge and current distributions, one naturally infers that these potentials also present suitably interpreted physical field entities mutually related to their sources. This way of thinking proved to be, from the physical point of view, very fruitful, having brought about a new vacuum field theory approach [68, 69, 70, 11] to alternative explaining the nature of the fundamental Maxwell equations and related electrodynamic phenomena.

We start from detailed revisiting the classical Ampere law in electrodynamics and show that main inferences suggested by physicists of the former centuries can be strongly extended to agree more exactly with many modern theoretical achievements and experimental results concerning the fundamental relationship of electrodynamic phenomena with the physical structure of vacuum as their principal carrier.

The important theoretical physical principles, characterizing the related electrodynamical vacuum field structure and based on the least action principle we discuss subject to different charged point particle dynamics, based on the fundamental least action principle. In particular, the main classical relativistic relationships, characterizing the charge point particle dynamics, are obtained by means of the least action principle within the Feynman's approach to the Maxwell electromagnetic equations and the Lorentz type force derivation. Moreover, for each least action principle constructed in the work, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws.

1.2. The Ampere law in electrodynamics - the classical and modified Lorentz forces derivations

The classical ingenious Andre-Marie Ampere's analysis of magnetically interacting to each other two electric currents in thin conductors, as is well known, was based [30, 50, 64, 83] on the following experimental fact: the force between two electric currents depends on the distance between conductors, their mutual spatial orientation and the quantitive values of currents. Having additionally accepted the infinitesimal superposition principle for this force A.M. Ampere had derived a general analytical force expression for the force between two infinitesimal elements of currents under regard:

$$df(r,r') = I\,I'\frac{(r-r')}{|r-r'|^2}\,\alpha(s,s';n)\,dl\,dl', \tag{1}$$

where vectors $r, r' \in \mathbb{E}^3$ point at infinitesimal currents $dr = s\,dl$, $dr' = s'\,dl'$ with normalized orientation vectors $s, s' \in \mathbb{E}^3$ of two closed conductors l and l' carrying currents $I \in \mathbb{R}$ and $I' \in \mathbb{R}$, respectively, the unit vector $n := (r-r')/|r-r',|$ which fixes the local spatial orientation of these conductors, the function $\alpha : (\mathbb{S}^2)^2 \times \mathbb{S}^2 \to \mathbb{R}$ is some real-valued smooth mapping. Taking further into account the mutual symmetry between the infinitesimal elements of currents dl and dl', belonging respectively to these two electric conductors, the infinitesimal force (1) was assumed by A.M. Ampere *to satisfy locally the third Newton's law*:

$$df(r,r') = -df(r',r) \tag{2}$$

with the mapping

$$\alpha(s,s';n) = \frac{\mu_0}{4\pi}\big(3k\langle s,n\rangle\langle s',n\rangle + k_2\ \langle s,s'\rangle\big), \tag{3}$$

where $\langle\cdot,\cdot\rangle$ is the natural scalar product in $\mathbb{E}^3$ and $k_1, k_2 \in \mathbb{R}$ are some still undetermined real and dimensionless parameters. The assumption (2) is evidently looking very restrictive and can be considered as reasonable only subject to a stationary system of conductors under regard, when the mutual action at a distance principle [30, 50] can be applied. Citing J.C. Maxwell [19]: "...we may draw the conclusions, first, that action and reaction are not always equal and opposite, and second, that apparatus may be constructed to generate any amount of work from its own resources. For let two oppositely electrified bodies A and B travel along the line joining them with equal velocities in the direction AB, then if either the potential or the attraction of the bodies at a given time is that due to their position at some former time (as these authors suppose), B, the foremost body, will attract A forwards more than B attracts A backwards. Now let A and B be kept asunder by a rigid rod. The combined system, if set in motion in the direction AB, will pull in that direction with a force which may either continually augment the velocity, or may be used as an inexhaustible source of energy."

Based on the fact that there is no possibility to measure the force between two infinitesimal current elements, A.M. Ampere took into account (2), (3) and calculated the corresponding force exerted by the whole conductor l' on an infinitesimal current element of the other conductor:

$$\begin{aligned} dF(r) &:= \oint_{l'} df(r,r') = \\ &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'} \tfrac{r-r'}{|r-r'|^2}\big[3k_1\langle dr, \tfrac{r-r'}{|r-r'|}\rangle\langle dr', \tfrac{r-r'}{|r-r'|}\rangle + k_2\tfrac{r-r'}{|r-r'|}\langle dr, dr'\rangle\big] = \\ &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'} \nabla_{r'}\big(\tfrac{1}{|r-r'|}\big)\big[3k_1\langle dr, r-r'\rangle\langle dr', r-r'\rangle + k_2\langle dr, dr'\rangle\big], \end{aligned}$$

which can be equivalently transformed as

$$\begin{aligned} dF(r) &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'} \nabla_{r'}(\tfrac{1}{|r-r'|})\big[3k_1\langle dr, r-r'\rangle\langle dr', r-r'\rangle + k_2\langle dr, dr'\rangle\big] = \\ &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'} \nabla_{r'}(\tfrac{1}{|r-r'|})\big[k_1(3\langle dr, r-r'\rangle\langle dr', r-r'\rangle - \langle dr, dr'\rangle) + \\ &+ (k_1+k_2)\langle dr, dr'\rangle\big] = -k_1\tfrac{\mu_0 I}{4\pi}\big\langle dr, \nabla\oint_{l'} \tfrac{I' dr'}{|r-r'|}\big\rangle - (k_1+k_2)\nabla\oint_{l'}\big\langle dr, \tfrac{I' dr'}{|r-r'|}\big\rangle, \end{aligned} \tag{4}$$

owing to the integral identity

$$\oint_{l'} \nabla_{r'}\big(\tfrac{1}{|r-r'|}\big)\big[3\langle dr, r-r'\rangle\langle dr', r-r'\rangle - \langle dr, dr'\rangle\big] = \langle dr, \nabla\rangle\oint_{l'} \tfrac{dr'}{|r-r'|},$$

which can be easily checked by means of integration by parts. Introducing the vector potential

$$A(r) := \frac{\mu_0 I'}{4\pi} \oint_{l'} \frac{dr'}{|r - r'|},$$

generated by the conductor l' at point $r \in \mathbb{E}^3$ that belongs to the infinitesimal element dl of the conductor l, we can write the resulting infinitesimal force (4) in the following form:

$$\begin{aligned} dF(r) &= k_1[-I\ \langle dr, \nabla\rangle A(r) + I\ \nabla\langle dr, A(r)\rangle] - (2k_1 + k_2)I\ \nabla\langle dr, A(r)\rangle = \\ &= k_1 I dr \times (\nabla \times A(r)) - (2k_1 + k_2)I\ \nabla\langle dr, A(r)\rangle = \\ &= k_1 J(r)d^3r \times B(r) - (2k_1 + k_2)\nabla\langle J d^3r, A(r)\rangle, \end{aligned} \tag{5}$$

where we have taken into account the standard magnetic field definition $B(r) := \nabla \times A(r)$ and the corresponding current density relationship $J(r)d^3r := Idr$. There are, evidently, many different possibilities to choose the dimensionless parameters $k_1, k_2 \in \mathbb{R}$. In his analysis A.M. Ampere had chosen the case when $k_1 = 1, k_2 = -2$ and obtained the well known nowadays *magnetic force* expression

$$dF(r) = J(r)d^3r \times B(r),$$

which easily reduces to the *classical Lorentz expression*

$$df_L(r) = \xi u \times B(r)$$

for a force exerted by an external magnetic field on a moving with a constant velocity $u \in T(\mathbb{R}^3)$ point particle with an electric charge $\xi \in \mathbb{R}$.

If to take *an alternative choice* and put $k_1 = 1, k_2 = -1$, the expression (5) yields *a modified magnetic Lorentz type force*, exerted by an external magnetic field generated by a moving charged particle with a velocity $u' \in T(\mathbb{R}^3)$ on a point particle, endowed with the electric charge $\xi \in \mathbb{R}$ and moving with a velocity $u \in T(\mathbb{R}^3)$:

$$dF(r) = J(r)d^3r \times B(r) - \nabla\langle J(r)d^3r, A(r)\rangle, \tag{6}$$

which was before occasionally discussed in different works [55, 56, 62, 67, 74] and recently obtained and analyzed in details from the Lagrangian point of view in works [13, 68, 69, 70] in the following equivalent infinitesimal form:

$$\delta f_L(r) = \xi u \times (\nabla \times \xi\delta A(r)) - \xi\nabla\ \langle u - u_f, \delta A(r)\rangle, \tag{7}$$

where $\delta A(r) \in T^*(\mathbb{R}^3)$ denotes the magnetic potential generated by an external charged point particle moving with velocity $u_f \in T(\mathbb{R}^3)$ and exerting the magnetic force $\delta f_L(r)$ on the charged particle located at point $r \in \mathbb{R}^3$ and moving with velocity $u \in T(\mathbb{R}^3)$ with respect to a common reference system $\mathcal{K}$. We also need to mention here that the modified Lorentz force expression (6) does not take naturally into account the resulting pure electric force as the conductors l and l' are considered to be electrically neutral. Simultaneously, we see that the magnetic potential has a physical significance in its own right [13, 21, 55, 62, 74] and has meaning in a way that extends beyond the calculation of force fields.

Really, to obtain the Lorentz type force (6) exerted by the external magnetic field generated by *the whole conductor* l' on an infinitesimal current element dl of the conductor l, it is necessary to integrate the expression (7) along this conductor loop l':

$$
\begin{aligned}
& dF(r) := \oint_{l'} \delta f_L(r) = \\
& = J(r)dr \times (\nabla \times \oint_{l'} \delta A(r)) - \nabla \langle J(r)dr, \oint_{l'} \delta A(r)\rangle + \nabla \oint_{l'} \langle u', \xi \delta A(r)\rangle = \\
& = J(r)dr \times (\nabla \times A(r)) - \nabla \langle J(r)dr, \oint_{l'} \delta A(r)\rangle \nabla \oint_{l'} \langle dr', \xi \delta A(r)/dt\rangle = \\
& = J(r)dr \times B(r) - \nabla \langle J(r)dr, \oint_{l'} \delta A(r)\rangle + \nabla \int_{S(l')} \langle dS(l'), \nabla \times \xi \delta A(r)/dt\rangle = \\
& = J(r)dr \times B(r) - \nabla \langle J(r)dr, \oint_{l'} \delta A(r)\rangle + \nabla \oint_{l'} \langle dS(l'), \xi \delta B(r)/dt\rangle = \\
& = J(r)dr \times B(r) - \nabla \langle J(r)dr, \oint_{l'} \delta A(r)\rangle + \xi \nabla (d\Phi(r)/dt) = \\
& = J(r)dr \times B(r) - \nabla \langle J(r)dr, A(r)\rangle - \rho(r) d^3 r \nabla \overline{W} = \\
& = J(r)dr \times B(r) - \nabla \langle J(r)dr, \oint_{l'} \delta A(r)\rangle + \rho(r) d^3 r (-\nabla \overline{W} - \partial A(r)/\partial t) = \\
& = J(r)dr \times B(r) - \nabla \langle J(r)dr, \oint_{l'} \delta A(r)\rangle + \rho(r) d^3 r E(r),
\end{aligned}
$$

and obtain the equality

$$dF(r) = \rho(r)d^3 r E(r) + J(r)d^3 r \times B(r) - \nabla \langle J(r)d^3 r, A(r)\rangle, \tag{8}$$

where, by definition, the electric field $E(r) := -\nabla \overline{W} - \partial A(r)/\partial t$. Now one can easily derive from (8) *the Lorentz type force* expression

$$dF(r) = J(r)d^3 r \times B(r) - \nabla \langle J(r)d^3 r, A(r)\rangle,$$

taking into account that the whole electric field $E(r) = 0$, owing to the facts that the conductors are neutral and that the electric potential $\overline{W}(r)$ is a constant function of the spatial variable $r \in \mathbb{E}^3$.

The presented above analysis of the A.M. Ampere's derivation of the magnetic force expression (5), as well as its consequences (6) and (7) make it possible to suppose that the missed modified Lorentz type force expression (6) could also be embedded into the classical relativistic Lagrangian and related Hamiltonian formalisms, giving rise to eventually new aspects and interpretations of many observed during the past centuries "strangely" looking experimental phenomena.

In our investigation, we were in part inspired by works [18, 20, 26, 87] and especially by [33, 34, 73] to solving the classical problem of reconciling gravitational and electrodynamic charges within the Feynman proper time and zero energy point paradigms. First, we will revisit the classical Mach-Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (29) and (30), making use of the fundamental Lagrangian and Hamiltonian formalisms which were specially devised in [12, 70].

1.3. Classical Maxwell equations and their electromagnetic potentials revised

As the classical Lorentz force expression with respect to an arbitrary inertial reference frame is related with many theoretical and experimental controversies, such as the relativistic potential energy impact into the charged point particle mass, the Aharonov-Bohm effect [2] and the Abraham-Lorentz-Dirac radiation force [21, 41, 50] expression, the analysis of its structure subject to the assumed vacuum field medium structure is a very interesting and important problem, which was discussed by many physicists including E. Fermi, G. Schott, R. Feynman, F. Dyson [27, 28, 29, 30, 35, 79] and many others. To describe the essence of the electrodynamic problems related with the description of a charged point particle dynamics under external electromagnetic field, let us begin with analyzing the classical Lorentz force expression

$$\frac{dp}{dt} = F_L := \xi E + \xi u \times B, \tag{9}$$

where $\xi \in \mathbb{R}$ is a particle electric charge, $u \in T(\mathbb{R}^3)$ is its velocity vector [1, 10], expressed here in the light speed c units,

$$E := -\partial A/\partial t - \nabla\varphi \tag{10}$$

is the corresponding external electric field and

$$B := \nabla \times A \tag{11}$$

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector $A : M^4 \to \mathbb{E}^3$ and scalar $\varphi : M^4 \to \mathbb{R}$ potentials. Here, as before, the sign "∇" is the standard gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$, "$\times$" is the usual vector product in three-dimensional Euclidean vector space $\mathbb{E}^3 := (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$, which is naturally endowed with the classical scalar product $\langle\cdot,\cdot\rangle$. These potentials are defined on the Minkowski space $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$, which models a chosen laboratory reference frame $\mathcal{K}$. Now, it is a well known fact [30, 50, 64, 83] that the force expression (9) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge $\xi \to 0$. This also means that expression (9) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in classical manuals [30, 50]. As the classical Lorentz force expression (9) is a natural consequence of the interaction of a charged point particle with an ambient electromagnetic field, its corresponding derivation based on the general principles of dynamics, was deeply analyzed by R. Feynman and F. Dyson [27, 28, 30].

Taking this into account, it is natural to reanalyze this classical problem taking into account the Maxwell-Faraday wave theory aspect and specifying the corresponding vacuum field medium. Other questionable inferences from the classical electrodynamics theory, which strongly motivated the analysis in this work, are related both with an alternative interpretation of the well-known *Lorenz condition*, imposed on the four-vector of electromagnetic observable potentials $(\varphi, A) : M^4 \to T^*(M^4)$ and the classical Lagrangian formulation [50] of charged particle dynamics under external electromagnetic field. The Lagrangian approach is strongly dependent on an important Einsteinian notion of the proper reference frame $\mathcal{K}_\tau$ and the related least action principle, so before explaining it in more detail, we first to analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider with respect to a laboratory reference frame $\mathcal{K}$ an additional *Lorenz condition*

$$\frac{\partial \varphi}{\partial t} + \langle \nabla, A \rangle = 0, \tag{12}$$

a priori assumed the Lorentz invariant wave scalar field equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \rho \tag{13}$$

and the charge continuity equation

$$\frac{\partial \rho}{\partial t} + \langle \nabla, J \rangle = 0, \tag{14}$$

where $\rho : M^4 \to \mathbb{R}$ and $J : M^4 \to \mathbb{E}^3$ are, respectively, the charge and current densities of the ambient matter. Then one can derive [68, 70] that the Lorentz invariant wave equation

$$\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = J \tag{15}$$

and the classical electromagnetic Maxwell field equations [30, 41, 50, 64, 83]

$$\begin{aligned} \nabla \times E + \partial B/\partial t &= 0, \quad \langle \nabla, E \rangle = \rho, \\ \nabla \times B - \partial E/\partial t &= J, \quad \langle \nabla, B \rangle = 0, \end{aligned} \tag{16}$$

hold for all $(t, r) \in M^4$ with respect to the chosen laboratory reference frame $\mathcal{K}$. As shown by O.D. Jefimenko [42, 45], the corresponding solutions to (16) for the electric $E : M^4 \to \mathbb{E}^3$ and magnetic $B : M^4 \to \mathbb{E}^3$ fields can be represented by means of the following causally independent to each other field expressions

$$\begin{aligned} E(t,r) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\Big[\big(\frac{\rho(t_r,r')}{|r-r'|^3} + \frac{1}{|r-r'|^2 c}\frac{\partial\rho(t_r,r')}{\partial t}\big)(r-r') - \frac{1}{|r-r'|^2}\frac{\partial J(t_r,r')}{\partial t}\Big]d^3r', \\ B(t,r) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\big[\frac{J(t_r,r')}{|r-r'|^3} + \frac{1}{|r-r'|^2 c}\frac{\partial J(t_r,r')}{\partial t}\big] \times (r-r')d^3r', \end{aligned} \tag{17}$$

where $(t, r) \in M^4$ and $t_r = t - \frac{|r-r'|}{c}$ is the retarded time. The result (17) was based on direct derivation from the classical Lienard-Wiechert potentials [41, 45] solving the field equations (13) and (15), causally depending on the corresponding charge and current distributions. Based strongly on this fact in [42, 45] there was argued from physical point of view that related with equations (13) and (15) electric and magnetic potentials really constitute some suitably interpreted physical entities, in contrast to the usual statements [41, 30, 50] about their pure mathematical origin.

It is worth to notice here that, inversely, Maxwell's equations (16) do not directly reduce, via definitions (10) and (11), to the wave field equations (13) and (15) without the Lorenz condition (12). This fact and reasonings presented above are very important: they suggest that, when it comes to choose main governing equations, it proves to be natural replacing the Maxwell's equations (16) with the electric potential field equation (13), the Lorenz condition (12) and the charge continuity equation (14). To make the equivalence statement, claimed above, more transparent we formulate it as the following proposition.

Proposition 1.1. *The Lorentz invariant wave equation (13) together with the Lorenz condition (12) for the observable potentials* $(\varphi, A) : M^4 \to T^*(M^4)$ *and the charge continuity relationship (14) are completely equivalent to the Maxwell field equations (16).*

Proof. Substituting (12), into (13), one easily obtains

$$\frac{\partial^2 \varphi}{\partial t^2} = -\langle \nabla, \partial A/\partial t \rangle = \langle \nabla, \nabla\varphi \rangle + \rho,$$

which implies the gradient expression

$$\langle \nabla, -\partial A/\partial t - \nabla\varphi \rangle = \rho. \tag{18}$$

Taking into account the electric field definition (10), expression (18) reduces to $\langle \nabla, E \rangle = \rho$, which is the second of the first pair of Maxwell's equations (16).

Now upon applying $\nabla\times$ to definition (10), we find, owing to definition (11), that

$$\nabla \times E + \partial B/\partial t = 0,$$

which is the first pair of the Maxwell equations (16). Differenting the equation (13) by the temporal variable $t \in \mathbb{R}$ and taken into account the charge continuity equation (14), one finds that $\langle \nabla, \partial^2 A/\partial t^2 - \nabla^2 A - J \rangle = 0$. The latter is equivalent to the wave equation (15), which follows from the observation that the current vector $J : M^4 \to \mathbb{E}^3$ is defined by means of the charge continuity equation (14) up to a vector function $\nabla \times S : M^4 \to \mathbb{E}^3$. Now applying operation $\nabla\times$ to the definition (11), owing to the wave equation (15) one obtains the equation

$$\begin{aligned}\nabla \times B &= \nabla \times (\nabla \times A) = \nabla\langle \nabla, A\rangle - \nabla^2 A = \\ &= -\nabla(\partial\varphi/\partial t) - \partial^2 A/\partial t^2 + (\partial^2 A/\partial t^2 - \nabla^2 A) = \\ &= \frac{\partial}{\partial t}(-\nabla\varphi - \partial A/\partial t) + J = \partial E/\partial t + J,\end{aligned}$$

which leads directly to $\nabla \times B = \partial E/\partial t + J$, which is the first of the second pair of the Maxwell equations (16). The final "*no magnetic charge*" equation $\langle \nabla, B\rangle = \langle \nabla, \nabla \times A\rangle = 0$ in (16) follows directly from the elementary identity $\langle \nabla, \nabla\times\rangle = 0$, thereby completing the proof. □

This proposition allows us to consider the observable potential functions $(\varphi, A) : M^4 \to T^*(M^4)$ as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles imbedded in space-time M^4. As there was written by J.K. Maxwell [19]: "The conception of such a quantity, on the changes of which, and not on its absolute magnitude, the induction currents depends, occurred to Faraday at an early stage of his researches. He observed that the secondary circuit, when at rest in an electromagnetic field which remains of constant intensity, does not show any electrical effect, whereas, if the same state of the field had been suddenly produced, there would have been a current. Again, if the primary circuit is removed from the field, or the magnetic forces abolished, there is a current of the opposite kind. He therefore recognized in the secondary circuit, when in the electromagnetic field, a "peculiar electrical condition of matter" to which he gave the name of Electrotonic State." The following observation provides a strong support of this reasonings within this vacuum field theory approach:

Observation. *The Lorenz condition (12) actually means that the scalar potential field $\varphi\colon M^4 \to \mathbb{R}$ continuity relationship, whose origin lies in some new field conservation law, characterizes the deep intrinsic structure of the vacuum field medium.*

To make this observation more transparent and precise, let us recall the definition [50, 64, 30, 83] of the electric current $J : M^4 \to \mathbb{E}^3$ in the dynamical form

$$J := \rho u, \tag{19}$$

where the vector $u \in T(\mathbb{R}^3)$ corresponds to the charge velocity. Thus, the following continuity relationship

$$\partial\rho/\partial t + \langle \nabla, \rho u \rangle = 0 \tag{20}$$

holds, which can easily be rewritten [57] as the integral conservation law

$$\frac{d}{dt}\int_{\Omega_t} \rho(t,r) d^3 r = 0$$

for the charge inside of any bounded domain $\Omega_t \subset \mathbb{E}^3$, moving in the space-time M^4 with respect to the natural evolution equation $dr/dt := u$. Following the above reasoning, we obtain the following result.

Proposition 1.2. *The Lorenz condition (12) is equivalent to the integral conservation law*

$$\frac{d}{dt}\int_{\Omega_t} \varphi(t,r) d^3 r = 0, \tag{21}$$

where $\Omega_t \subset \mathbb{E}^3$ is any bounded domain, moving with respect to the charged point particle ξ evolution equation

$$dr/dt = u(t,r), \tag{22}$$

which represents the velocity vector of related local potential field changes propagating in the Minkowski space-time M^4. Moreover, for a particle with the distributed charge density $\rho : M^4 \to \mathbb{R}$, the following Umov type local energy conservation relationship holds for any $t \in \mathbb{R}$:

$$\frac{d}{dt}\int_{\Omega_t} \frac{\rho(t,r)\varphi(t,r)}{\sqrt{1-|u(t,r)|^2}} d^3 r = 0. \tag{23}$$

Proof. Consider first the corresponding solutions to potential field equations (13), taking into account condition (19). Owing to the standard results from [30, 50], one finds that $A = \varphi u$, which gives rise to the following form of the Lorenz condition (12):

$$\partial\varphi/\partial t + \langle \nabla, \varphi u \rangle = 0, \tag{24}$$

This obviously can be rewritten [57] as the integral conservation law (21), so the expression (21) is stated.

To state the local energy conservation relationship (23) it is necessary to combine the conditions (20), (24) and find that

$$\partial(\rho\varphi)/\partial t + \langle u, \nabla(\rho\varphi)\rangle + 2\rho\varphi\langle\nabla, u\rangle = 0. \tag{25}$$

Taking into account that the infinitesimal volume transformation $d^3r = \chi(t,r)d^3r_0$ for the Jacobian $\chi(t,r) := |\partial r(t;r_0)/\partial r_0|$ of the corresponding transformation $r : \Omega_{t_0} \to \Omega_t$, induced by the Cauchy problem for the differential relationship (22) for $t \in \mathbb{R}$ satisfies the evolution equation $d\chi/dt = \langle\nabla, u\rangle\chi$, easily following from (22), and applying to the equality (25) the operator $\int_{\Omega_{t_0}}(...)\chi^2 d^3r_0$, one obtains the equality

$$\begin{aligned}0 = \int_{\Omega_{t_0}} \tfrac{d}{dt}(\rho\varphi\chi^2)d^3r_0 = \frac{d}{dt}\int_{\Omega_{t_0}}(\rho\varphi\chi)Jd^3r_0 = \\ = \frac{d}{dt}\int_{\Omega_t}(\rho\varphi\chi\ )d^3r \ := \ \tfrac{d}{dt}\mathcal{E}(\xi;\Omega_t).\end{aligned} \tag{26}$$

Here we denote by $\xi := \int_{\Omega_t}\rho(t,r)d^3r$ the conserved charge and by $\mathcal{E}(\xi;\Omega_t) := \int_{\Omega_t}(\rho\varphi\chi)d^3r$ the conservation quantity of local energy. The latter quantity can be simplified, owing to the infinitesimal Lorentz invariance four-volume measure relationship $d^3r(t,r_0)\wedge dt = d^3r_0 \wedge dt_0$, where variables $(t,r) \in \mathbb{R}_t \times \Omega_t \subset M^4$ are, within the present context, taken with respect to the moving reference frame $\mathcal{K}_t$, related to the infinitesimal charge quantity $d\xi(t,r) := \rho(t,r)d^3r$, and variables $(t_0,r_0) \in \mathbb{R}_{t_0} \times \Omega_{t_0} \subset M^4$ are taken with respect to the laboratory reference frame $\mathcal{K}_{t_0}$, related to the infinitesimal charge quantity $d\xi(t_0,r_0) = \rho(t_0,r_0)d^3r_0$, satisfying the charge conservation invariance $d\xi(t,r) = d\xi(t_0,r_0)$. Using the mentioned above infinitesimal Lorentz invariance relationships one can calculate the local energy conservation quantity $\mathcal{E}(\xi;\Omega_0)$ as

$$\begin{aligned}\mathcal{E}(\xi;\Omega_0) &= \int_{\Omega_t}(\rho\varphi\chi)d^3r = \int_{\Omega_t}\big(\rho\varphi\tfrac{d^3r}{d^3r_0}\big)d^3r = \\ &= \int_{\Omega_t}\big(\rho\varphi\tfrac{d^3r\wedge dt}{d^3}r_0\wedge dt\big)d^3r = \int_{\Omega_t}\Big(\rho\varphi\frac{d^3r_0\wedge dt_0}{d^3r_0\wedge dt}\Big)d^3r = \\ &= \int_{\Omega_t}(\rho\varphi\frac{dt_0}{dt})d^3r = \int_{\Omega_t}\frac{\rho\varphi\, d^3r}{\sqrt{1-|u|^2}}\end{aligned} \tag{27}$$

where we took into account that $dt = dt_0\sqrt{1-|u|^2}$. Thus, owing to (26) and (27) the local energy conservation relationship (23) is satisfied, proving the proposition. □

The constructed above local energy conservation quantity (27) can be rewritten as

$$\mathcal{E}(\xi;\Omega_t) = \int_{\Omega_t} \frac{d\xi(t,r)\varphi(t,r)}{\sqrt{1-|u|^2}} := \int_{\Omega_t} d\mathcal{E}(t,r)$$

where $d\mathcal{E}(t,r) = d\xi(t,r)\varphi(t,r)/\sqrt{1-|u|^2}$ is the distributed in vacuum electromagnetic field energy density, related with the electric charge $d\xi(t,r)$ located at a point $(t,r) \in M^4$.

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called *the vacuum potential field*, which determines the observable interactions between charged point particles. More precisely, we can *a priori* endow the ambient vacuum medium with a scalar potential energy field density function $W := \xi\varphi : M^4 \to \mathbb{R}$, where $\xi \in \mathbb{R}_+$ is the value of an elementary charge quantity, satisfying the governing *vacuum field equations*

$$\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = \rho\xi, \quad \frac{\partial W}{\partial t} + \langle \nabla, \breve{A}\rangle = 0, \quad \frac{\partial^2 \breve{A}}{\partial t^2} - \nabla^2 \breve{A} = \xi\rho v, \quad \breve{A} = Wv, \tag{28}$$

taking into account the external charged sources, which possess a virtual capability for disturbing the vacuum field medium. Moreover, this vacuum potential field function $W : M^4 \to \mathbb{R}$ allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles, interacting through the gravity.

The latter leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field $\bar{W}: M^4 \to \mathbb{R}$, assigned to a charged point particle moving in the vacuum field medium with velocity $u \in T(\mathbb{R}^3)$ and located at point $r(t) = R(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$. As can be readily shown [68, 69, 70, 74], the corresponding evolution equation governing the related potential field function $\bar{W}: M^4 \to \mathbb{R}$, assigned to a moving in the space $\mathbb{E}^3$ charged particle ξ under the stationarily distributed field sources, has the form

$$\frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W}, \tag{29}$$

where $\bar{W} := W(t,r)|_{r\to R(t)}$, $u(t) := dR(t)/dt$ at point particle location $(t, R(t)) \in M^4$.

Similarly, if there are two interacting charged point particles located at points $r(t) = R(t)$ and $r_f(t) = R_f(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$ and moving with velocities $u := dR(t)/dt$ and $u_f := dR_f(t)/dt$, respectively, then the corresponding potential field function $\bar{W}'^4 \to \mathbb{R}$, considered with respect to the reference frame $\mathcal{K}'$ specified by Euclidean coordinates $(t', r - r_f) \in \mathbb{E}^4$ and moving with the velocity $u_f \in T(\mathbb{R}^3)$

subject to the laboratory reference frame $\mathcal{K}$, should satisfy [68, 69] with respect to the reference frame $\mathcal{K}'$ the dynamical equality

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla \bar{W}', \tag{30}$$

where $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$, are the velocity vectors. The latter comes with respect to the laboratory reference frame $\mathcal{K}$ about the dynamical equality

$$\frac{d}{dt}[-\bar{W}\ (u\ - u_f)] = -\nabla \bar{W}(1 - |u_f|^2). \tag{31}$$

The dynamical potential field equations (29) and (30) appear to have important properties and can be used as means for representing classical electrodynamic phenomena. Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of moving charged point particles in an external electromagnetic field.

1.3.1. Classical relativistic electrodynamics revisited

The classical relativistic electrodynamics of a freely moving charged point particle in the Minkowski space-time $M^4 := \mathbb{R} \times \mathbb{E}^3$ is based on the Lagrangian approach. It uses the Lagrangian function

$$\mathcal{L} := -m_0\sqrt{1 - |u|^2}, \tag{32}$$

where $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass and $u \in T(\mathbb{R}^3)$ is its spatial velocity in the Euclidean space $\mathbb{E}^3$, expressed here and in the sequel in light speed units (with light speed c). The least action principle in the form

$$\delta S = 0, \quad S := -m_0 \int_{t_1}^{t_2} \sqrt{1 - |u|^2} dt$$

for any fixed temporal interval $[t_1, t_2] \subset \mathbb{R}$ gives rise to the well-known relativistic relationships for the mass of the particle

$$m = \frac{m_0}{\sqrt{1 - |u|^2}}, \tag{33}$$

the momentum of the particle

$$p := mu = \frac{m_0 u}{\sqrt{1 - |u|^2}}$$

and the energy of the particle

$$\mathcal{E}_0 = m = \frac{m_0}{\sqrt{1 - |u|^2}}.$$

It follows from [50, 64], that the origin of the Lagrangian (32) can be extracted from the action

$$S := -m_0 \int_{t_1}^{t_2} \sqrt{1-|u|^2}dt = -m_0 \int_{\tau_1}^{\tau_2} d\tau, \tag{34}$$

on the suitable temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$, where, by definition,

$$d\tau := dt\sqrt{1-|u|^2}$$

and $\tau \in \mathbb{R}$ is the so-called, proper temporal parameter assigned to a freely moving particle with respect to the rest reference frame $\mathcal{K}_r$. The action (34) is rather questionable from the dynamical point of view, since it is physically defined with respect to the rest reference frame $\mathcal{K}_r$, giving rise to the constant action $S = -m_0(\tau_2 - \tau_1)$, as the limits of integrations $\tau_1 < \tau_2 \in \mathbb{R}$ were taken to be fixed from the very beginning. Moreover, considering this particle to have charge $\xi \in \mathbb{R}$ and be moving in the Minkowski space-time M^4 under action of an electromagnetic field $(\varphi, A) \in \mathbb{R} \times \mathbb{E}^3$, the corresponding classical (relativistic) action functional is chosen (see [12, 30, 50, 64, 70, 83]) as follows:

$$S := \int_{\tau_1}^{\tau_2} [-m_0 d\tau + \xi\langle A, \dot{r}\rangle d\tau - \tfrac{\xi\varphi}{\sqrt{1-|u|^2}} d\tau], \tag{35}$$

with respect to the *proper reference system*, parameterized by the Euclidean space-time variables $(\tau, r) \in \mathbb{E}^4$, where we have denoted $\dot{r} := dr/d\tau$ in contrast to the definition $u := dr/dt$. The action (35) can be rewritten with respect to the laboratory reference frame $\mathcal{K}$ moving with velocity vector $u \in \mathbb{E}^3$ as

$$S = \int_{t_1}^{t_2} \mathcal{L}dt, \ \ \mathcal{L} := -m_0\sqrt{1-|u|^2} + \xi\,\langle A, u\rangle - \xi\varphi, \tag{36}$$

on the suitable temporal interval $[t_1, t_2] \subset \mathbb{R}$, which gives rise to the following [50, 30, 64, 83] dynamical expressions

$$P = p + \xi A, \quad p = mu, \quad m = \frac{m_0}{\sqrt{1-|u|^2}},$$

for the particle momentum and

$$\mathcal{E}_0 = \sqrt{m_0^2 + |P - \xi A|^2} + \xi\varphi \tag{37}$$

for the energy of charged particle ξ, where, by definition, $P \in \mathbb{E}^3$ is the common momentum of the particle and the ambient electromagnetic field at a space-time point $(t, r) \in M^4$.

The expression (37) for the particle energy $\mathcal{E}_0$ also appears open to question, since the potential energy $\xi\varphi$, entering additively, has no affect on the particle mass $m = m_0/\sqrt{1-|u|^2}$. This was noticed by L. Brillouin [17], who remarked that the fact that the potential energy has no affect on the particle mass tells us that "... any possibility of existence of a particle mass related with an external potential energy, is completely excluded". Moreover, it is necessary to stress here that the least action principle (36), formulated with respect to the laboratory reference frame $\mathcal{K}$ time parameter $t \in \mathbb{R}$, appears logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent reference frames. This was first mentioned by R. Feynman in [30], in his efforts to rewrite the Lorentz force expression with respect to the proper reference frame $\mathcal{K}_r$. This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past [16, 17, 30, 64, 86] and present [3, 8, 18, 20, 26, 32, 33, 34, 37, 38, 52, 53, 59, 60] and [9, 61, 62, 66, 75, 77, 82, 85] to try to develop alternative relativity theories based on completely different space-time and matter structure principles. Some of them prove to be closely related with a virtual relationship between electrodynamics and gravity, based on classical works of H. Lorentz, Schott, J. Schwinger, R. Feynman [30, 54, 79, 80] and many others on the so called "electrodynamic mass" of elementary particles. Arguing of that mass, one can readily come to a certain paradox: by the well-known energy-mass relationship, the particle mass determines the energy of its gravitational field. Yet this energy should lead to an increase in the mass of the particle that in turn should lead to increased gravitational field and so on. In the limit, for instance, an electron must have infinite mass and energy, which is not observed in reality.

There is also another controversial inference from the action expression (36). As one can easily show [30, 50, 64, 83], the corresponding dynamical equation for the Lorentz force is given by

$$dp/dt = F := \xi E + \xi u \times B. \tag{38}$$

We have defined here, as before, $E := -\partial A/\partial t - \nabla\varphi$ for the corresponding electric field and $B := \nabla \times A$ for the related magnetic field, acting on the charged point particle ξ. The expression (38) means, in particular, that the Lorentz force F depends linearly on the particle velocity vector $u \in T(\mathbb{R}^3)$, and so there is a strong dependence on the reference frame with respect to which the charged particle ξ moves. Attempts to reconcile this and some related controversies [17, 30, 48, 62, 74, 82] forced Einstein to elaborate his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe. Here we must mention that the classical Lagrangian function $\mathcal{L}$ in (36) is written in terms of a combination of terms expressed by means of both the Euclidean proper reference frame variables $(\tau, r) \in \mathbb{E}^4$ and arbitrarily chosen Minkowski reference frame variables $(t, r) \in M^4$.

These problems were recently analyzed using a completely different "geometry-free" approach [68, 69, 74], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the well known Lorentz transformations of the space-time reference frames with respect to which the action functional (36) is invariant. From this point of view, there are interesting for discussion conclusions in [5, 36, 40, 78, 88], in which some electrodynamic models, possessing intrinsic Galilean and Poincaré-Lorentz symmetries, were reanalyzed from diverse geometrical points of view. From completely different point of view the related electrodynamics of charged particles was reanalyzed in [42, 43, 44, 45, 46], where all relativistic relationships were successfully infered from the classical Lienard-Wiechert potentials, solving the corresponding electromagnetic equations. Subject to a possible geometric space-type structure and the related vacuum field background, exerting the decisive influence on the particle dynamics, we need to mention here recent works [4, 81] and the closely related with their ideas the classical articles [47, 65]. Next, we shall revisit the results obtained in [68, 69] from the classical Lagrangian and Hamiltonian formalisms [12] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and gravitational effects.

1.4. The vacuum field theory electrodynamics equations: Lagrangian analysis

1.4.1. A moving in vacuum point charged particle – an alternative electrodynamic model

In the vacuum field theory approach [68, 69] to combining electromagnetism and the gravity, the main vacuum potential field function $\bar{W} : M^4 \to \mathbb{R}$, related to a charged point particle ξ under the external stationarily distributed field sources, satisfies the dynamical equation (28), namely

$$\frac{d}{dt}(-\bar{W}u) = -\nabla \bar{W} \tag{39}$$

in the case when the external charged particles are at rest, where, as above, $u := dr/dt$ is the particle velocity with respect to some reference system.

To analyze the dynamical equation (39) from the Lagrangian point of view, we write the corresponding action functional as

$$S := -\int_{t_1}^{t_2} \bar{W} dt = -\int_{\tau_1}^{\tau_2} \bar{W} \sqrt{1 + |\dot{r}|^2}\, d\tau, \tag{40}$$

expressed with respect to the proper reference frame $\mathcal{K}_r$. Fixing the proper temporal parameters $\tau_1 < \tau_2 \in \mathbb{R}$, one finds from the least action principle ($\delta S = 0$) that

$$\begin{aligned} p &:= \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}/\sqrt{1+|\dot{r}|^2} = -\bar{W}u, \\ \dot{p} &:= dp/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}\sqrt{1+|\dot{r}|^2}, \end{aligned} \tag{41}$$

where, owing to (40), the corresponding Lagrangian function is

$$\mathcal{L} := -\bar{W}\sqrt{1+|\dot{r}|^2}. \tag{42}$$

Recalling now the definition of the particle mass

$$m := -\bar{W} \tag{43}$$

and the relationships $d\tau = dt\sqrt{1-|u|^2}$, $\dot{r}d\tau = udt$, from (41) we easily obtain exactly the dynamical equation (39). Moreover, one now readily finds that the dynamical mass, defined by means of expression (43), is given as $m = \frac{m_0}{\sqrt{1-|u|^2}}$, which coincides with the equation (33) of the preceding section. Now one can formulate the following proposition using the above results

Proposition 1.3. *The alternative freely moving point particle electrodynamic model (39) allows the least action formulation (40) with respect to the "rest" reference frame variables, where the Lagrangian function is given by expression (42). Its electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Subsection 1.3.1.*

1.4.2. A moving in vacuum interacting two charge system - an alternative electrodynamic model

We proceed now to the case when our charged point particle ξ moves in the space-time with velocity vector $u \in T(\mathbb{R}^3)$ and interacts with another external charged point particle ξ_f, moving with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to a common reference frame $\mathcal{K}$. As shown in [68, 69], the respectively modified dynamical equation for the vacuum potential field function $\bar{W}' : M^4 \to \mathbb{R}$ subject to the moving reference frame $\mathcal{K}'$ is given by equality (30), or

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}', \tag{44}$$

where, as before, $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$ are the velocity vectors. Since the external charged particle ξ_f moves in the space-time M^4, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potentials $A : M^4 \to \mathbb{E}^3$ and $A' : M^4 \to \mathbb{E}^3$ are defined, owing to the results of [68, 69, 74], as

$$\xi A := \bar{W}u_f, \quad \xi A' := \bar{W}'u'_f, \tag{45}$$

Whence, taking into account that the field potential

$$\bar{W} = \frac{\bar{W}'}{\sqrt{1-|u_f|^2}} \tag{46}$$

and the particle momentum $p' = -\bar{W}'u' = -\bar{W}u$, equality (44) becomes equivalent to

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \tag{47}$$

if considered with respect to the moving reference frame $\mathcal{K}'$, or to the Lorentz type force equality

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1-|u_f|^2), \tag{48}$$

if considered with respect to the laboratory reference frame $\mathcal{K}$, owing to the classical Lorentz invariance relationship (46), as the corresponding magnetic vector potential, generated by the external charged point test particle ξ_f with respect to the reference frame $\mathcal{K}'$, is identically equal to zero. To imbed the dynamical equation (48) into the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (40):

$$S := -\int_{\tau_1}^{\tau_2} \bar{W}' \sqrt{1+|\dot{r}-\dot{r}_f|^2}\, d\tau. \tag{49}$$

Here, as before, $\bar{W}'$ is the respectively calculated vacuum field potential $\bar{W}$ subject to the moving reference frame $\mathcal{K}'$, $\dot{r} = u'dt'/d\tau, \dot{r}_f = u'_f dt'/d\tau$, $d\tau = dt'\sqrt{1-|u'-u'_f|^2}$, taking into account the relative velocity of the charged point particle ξ subject to the reference frame $\mathcal{K}'$, specified by the Euclidean coordinates $(t', r-r_f) \in \mathbb{R}^4$, and moving simultaneously with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}$, specified by the Minkowski coordinates $(t, r) \in M^4$ and related to those of the reference frame $\mathcal{K}'$ and $\mathcal{K}_\tau$ by means of the following infinitesimal relationships:

$$dt^2 = (dt')^2 + |dr_f|^2, \;\; (dt')^2 = d\tau^2 + |dr - dr_f|^2. \tag{50}$$

So, it is clear in this case that our charged point particle ξ moves with the velocity vector $u' - u'_f \in T(\mathbb{R}^3)$ with respect to the reference frame $\mathcal{K}'$ in which the external charged particle ξ_f is at rest. Thereby, we have reduced the problem of deriving the charged point particle ξ dynamical equation to that solved in Subsection 1.3.1.

Now we can compute the least action variational condition $\delta S = 0$, taking into account that, owing to (49), the corresponding Lagrangian function with respect to the proper reference frame $\mathcal{K}_\tau$ is given by

$$\mathcal{L} := -\bar{W}'\sqrt{1 + |\dot{r} - \dot{r}_f|^2}. \tag{51}$$

As a result of simple calculations, the generalized momentum of the charged particle ξ equals

$$P := \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{-\bar{W}'\dot{r} + \bar{W}'\dot{r}_f}{\sqrt{1 + |\dot{r} - \dot{r}_f|^2}} = m'u' + \xi A' := p' + \xi A' = p + \xi A, \tag{52}$$

where, owing to (46) $p' := -\bar{W}'u' = -\bar{W}u = p \in \mathbb{E}^3$, $A' = \bar{W}'u'_f = \bar{W}u_f = A \in \mathbb{E}^3$, and giving rise to the dynamical equality

$$\frac{d}{d\tau}(p' + \xi A') = -\nabla \bar{W}'\sqrt{1 + |\dot{r} - \dot{r}_f|^2} \tag{53}$$

with respect to the proper reference frame $\mathcal{K}_\tau$. As $dt' = d\tau\sqrt{1 + |\dot{r} - \dot{r}_f|^2}$ and $\sqrt{1 + |\dot{r} - \dot{r}_f|^2} = (1 - |u' - u'_f|^2)^{-\frac{1}{2}}$, we obtain from (53) the equality

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \tag{54}$$

exactly coinciding with equality (47) subject to the moving reference frame $\mathcal{K}'$. Now, making use of expressions (50) and (46), one can rewrite (54) as that with respect to the laboratory reference frame $\mathcal{K}$:

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}' \Rightarrow \frac{d}{dt'}\Big(\frac{-\bar{W}u'}{\sqrt{1+|u'_f|^2}} + \frac{\xi \bar{W}u'_f}{\sqrt{1+|u'_f|^2}}\Big) = -\frac{\nabla \bar{W}}{\sqrt{1+|u'_f|^2}} \Rightarrow$$

$$\frac{d}{dt'}\Big(\frac{-\bar{W}dr}{\sqrt{1+|u'_f|^2}dt'} + \frac{\xi \bar{W}dr_f}{\sqrt{1+|u'_f|^2}}\Big) = -\frac{\nabla \bar{W}}{\sqrt{1+|u'_f|^2}} \Rightarrow$$

$$\frac{d}{dt}(-\bar{W}\frac{dr}{dt} + \xi \bar{W}\frac{dr_f}{dt}) = -\nabla \bar{W}(1 - |u_f|^2),$$

exactly coinciding with (48):

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1 - |u_f|^2). \tag{55}$$

Remark 1.4. The equation (55) allows to infer the following important and physically reasonable phenomenon: if the test charged point particle velocity $u_f \in T(\mathbb{R}^3)$ tends to the light velocity $c = 1$, the corresponding acceleration force $F_{ac} :=$

$-\nabla \bar{W}(1-|u_f|^2)$ is vanishing. Thereby, the electromagnetic fields, generated by such rapidly moving charged point particles, have no influence on the dynamics of charged objects if observed with respect to an arbitrarily chosen laboratory reference frame $\mathcal{K}$.

The latter equation (55) can be easily rewritten as

$$\begin{aligned}\frac{dp}{dt} &= -\nabla \bar{W} - \xi dA/dt + \nabla \bar{W}|u_f|^2 = \\ &= \xi\big(-\xi^{-1}\nabla \bar{W} - \partial A/\partial t\big) - \xi\langle u, \nabla\rangle A + \xi\nabla\langle A, u_f\rangle,\end{aligned}$$

or, using the well-known [50] identity

$$\nabla\langle a, b\rangle = \langle a, \nabla\rangle b + \langle b, \nabla\rangle a + b \times (\nabla \times a) + a \times (\nabla \times b), \tag{56}$$

where $a, b \in \mathbb{E}^3$ are arbitrary vector functions, in the standard Lorentz type form

$$\frac{dp}{dt} = \xi E + \xi u \times B - \nabla\langle \xi A, u - u_f\rangle. \tag{57}$$

The equality (57), being before found and written down with respect to the moving reference frame $\mathcal{K}'$ in [68, 69, 74] and with some inconsistency in [58] allows us to formulate the next important proposition.

Proposition 1.5. *The alternative classical relativistic electrodynamic model (47) allows the least action formulation based on the action functional (49) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Lagrangian function is given by expression (51). The resulting Lorentz type force expression equals (57), being modified by the additional force component $F_c := -\nabla\langle \xi A, u - u_f\rangle$, important for explanation [2, 15, 84] of the well known Aharonov-Bohm effect.*

1.4.3. A moving charged point particle dynamics formulation, dual to the classical relativistic invariant alternative electrodynamic model

It is easy to see that the action functional (49) is written utilizing the classical Galilean transformations of reference frames. If we now consider the action functional (40) for a charged point particle moving with respect the reference frame $\mathcal{K}_r$, and take into account its interaction with an external magnetic field generated by the vector potential $A\colon M^4 \to \mathbb{E}^3$, it can be naturally generalized as

$$S := \int_{t_1}^{t_2} (-\bar{W}dt + \xi\langle A, dr\rangle) = \int_{\tau_1}^{\tau_2} [-\bar{W}\sqrt{1+|\dot{r}|^2} + \xi\langle A, \dot{r}\rangle]d\tau, \tag{58}$$

where $d\tau = dt\sqrt{1-|u|^2}$.

Thus, the corresponding common particle-field momentum takes the form

$$P := \frac{\partial \mathcal{L}}{\partial \dot{r}} = -\frac{\bar{W}\dot{r}}{\sqrt{1+|\dot{r}|^2}} + \xi A = mu + \xi A := p + \xi A, \tag{59}$$

and satisfies

$$\dot{P} := \frac{dP}{d\tau} = \frac{\partial \mathcal{L}}{\partial r} = -\nabla \bar{W}\sqrt{1+|\dot{r}|^2} + \xi \nabla \langle A, \dot{r}\rangle = \frac{-\nabla \bar{W} + \xi \nabla \langle A, u\rangle}{\sqrt{1-|u|^2}}, \tag{60}$$

where

$$\mathcal{L} := -\bar{W}\sqrt{1+|\dot{r}|^2} + \xi \langle A, \dot{r}\rangle \tag{61}$$

is the corresponding Lagrangian function. Since $d\tau = dt\sqrt{1-|u|^2}$, one easily finds from (60) that

$$\frac{dP}{dt} = -\nabla \bar{W} + \xi \nabla \langle A, u\rangle. \tag{62}$$

Upon substituting (59) into (62) and making use of the identity (56), we obtain the classical expression for the Lorentz force F, acting on the moving charged point particle ξ :

$$\frac{dp}{dt} := F = \xi E + \xi u \times B, \tag{63}$$

where $E := -\xi^{-1}\nabla \bar{W} - \frac{\partial A}{\partial t}$ is its associated electric field and $B := \nabla \times A$ is the corresponding magnetic field. This result can be summarized as follows:

Proposition 1.6. *The classical relativistic Lorentz force (63) allows the least action formulation (58) with respect to the rest reference frame variables, where the Lagrangian function is given by formula (61). Yet its electrodynamics, described by the Lorentz force (63), is not equivalent to the classical relativistic moving point particle electrodynamics, described by means of the Lorentz force (38), as the inertial mass expression* $m = -\bar{W}$ *does not coincide with that of (33).*

Expressions (63) and (57) are equal up to the gradient term

$$F_c := -\xi \nabla \langle A, u - u_f\rangle,$$

which reconciles the Lorentz forces acting on a charged moving particle ξ with respect to different reference frames. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze both electromagnetic and gravitational fields simultaneously by employing the new definition of the dynamical mass by means of expression (43).

1.5. The vacuum field theory electrodynamics equations: Hamiltonian analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [1, 6, 39, 71, 83]. As we have already formulated our vacuum field theory of a moving charged particle ξ in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (40), (51) and (58).

Take, first, the Lagrangian function (42) and the momentum expression (41) for defining the corresponding Hamiltonian function with respect to the moving reference frame $\mathcal{K}_r$:

$$H := \langle p, \dot{r} \rangle - \mathcal{L} = -\frac{\langle p, p \rangle}{\bar{W}\sqrt{1-|p|^2/\bar{W}^2}} + \frac{\bar{W}}{\sqrt{1-|p|^2/\bar{W}^2}} = -\sqrt{\bar{W}^2-|p|^2}. \quad (64)$$

Consequently, it is easy to show [1, 6, 71, 83] that the Hamiltonian function (64) is a conservation law of the dynamical field equation (39), that is $\frac{dH}{d\tau} = \frac{dH}{dt} = 0$ for all $\tau, t \in \mathbb{R}$ which naturally leads to an energy interpretation of H. Thus, we can represent the particle energy as $\mathcal{E} = \sqrt{\bar{W}^2 - |p|^2}$. Accordingly the Hamiltonian equivalent to the vacuum field equation (39) can be written as

$$\dot{r} := \frac{dr}{d\tau} = \frac{\partial H}{\partial p} = \frac{p}{\sqrt{\bar{W}^2 - |p|^2}}, \quad \dot{p} := \frac{dp}{d\tau} = -\frac{\partial H}{\partial r} = \frac{\bar{W}\nabla\bar{W}}{\sqrt{\bar{W}^2 - |p|^2}}, \quad (65)$$

and we have the following result.

Proposition 1.7. *The alternative freely moving point particle electrodynamic model (39) allows the canonical Hamiltonian formulation (65) with respect to the "rest" reference frame variables, where the Hamiltonian function is given by expression (64). Its electrodynamics is completely equivalent to the classical relativistic freely moving point particle electrodynamics described in Subsection 1.3.1.*

In the analogous manner, one can now use the Lagrangian (51) to construct the Hamiltonian function for the dynamical field equation (47), describing the motion of charged particle ξ in an external electromagnetic field in the canonical Hamiltonian form:

$$\dot{r} := \frac{dr}{d\tau} = \frac{\partial H}{\partial P}, \quad \dot{P} := \frac{dP}{d\tau} = -\frac{\partial H}{\partial r}, \quad (66)$$

where

$$\begin{aligned} H &:= \langle P, \dot{r}\rangle - \mathcal{L} = \left\langle P, \dot{r}_f - \tfrac{P}{\sqrt{(\bar{W}')^2 - |P|^2}}\right\rangle + \tfrac{(\bar{W}')^2}{\sqrt{(\bar{W}')^2-|P|^2}} = \\ &= \langle P, \dot{r}_f\rangle + \tfrac{|P|^2}{\sqrt{(\bar{W}')^2-|P|^2}} - \tfrac{(\bar{W}')^2}{\sqrt{(\bar{W}')^2-|P|^2}} = \\ &- \tfrac{(\bar{W}')^2-|P|^2}{\sqrt{(\bar{W}')^2-|P|^2}} + \langle P, \dot{r}_f\rangle = -\sqrt{(\bar{W}')^2 - |P|^2} - \tfrac{\xi\langle A', P\rangle}{\sqrt{(\bar{W}')^2-|P|^2}} = \\ &- \sqrt{\bar{W}^2 - |\xi A|^2 - |P|^2} - \tfrac{\xi\langle A, P\rangle}{\sqrt{\bar{W}^2-|\xi A|^2-|P|^2}}, \end{aligned} \tag{67}$$

being written with respect to the laboratory reference frame $\mathcal{K}$. Here we took into account that, owing to definitions (45), (46) and (52),

$$\begin{aligned} \xi A' &:= \bar{W}'u_f' = \bar{W}'\frac{dr_f}{dt'} = \xi A = \bar{W}'\frac{dr_f}{d\tau}\cdot\frac{d\tau}{dt'} = \bar{W}'\dot{r}_f\sqrt{1-|u-u_f|^2} = \\ &= \frac{\bar{W}'\dot{r}_f}{\sqrt{1+|\dot{r}-\dot{r}_f|^2}} = -\dot{r}_f\sqrt{(\bar{W}')^2 - |P|^2}, \end{aligned}$$

and, in particular,

$$\dot{r}_f = -\frac{\xi A}{\sqrt{(\bar{W}'^2 - |P|^2}}, \quad \bar{W} = \frac{\bar{W}'}{\sqrt{1-|u_f|^2}},$$

where $A : M^4 \to \mathbb{R}^3$ is the related magnetic vector potential generated by the moving external charged particle ξ_f. Equations (66) can be rewritten with respect to the laboratory reference frame $\mathcal{K}$ in the form

$$\frac{dr}{dt} = u, \quad \frac{dp}{dt} = \xi E + \xi u \times B - \xi\nabla\langle A, u - u_f\rangle, \tag{68}$$

which coincides with the result (57).

Whence, we see that the Hamiltonian function (67) satisfies the energy conservation conditions $\frac{dH}{d\tau} = \frac{dH}{dt'} = \frac{dH}{dt} = 0$, for all τ, t' and $t \in \mathbb{R}$, and that the suitable energy expression is

$$\mathcal{E} = \sqrt{\bar{W}^2 - \xi^2|A|^2 - |P|^2} + \frac{\xi\langle A, P\rangle}{\sqrt{\bar{W}^2 - \xi^2|A|^2 - |P|^2}}, \tag{69}$$

where the generalized momentum $P = p + \xi A$. The result (69) differs essentially from that obtained in [50], which makes use of the Einsteinian Lagrangian for a moving charged point particle ξ in an external electromagnetic field. Thus, we obtain the following proposition:

Proposition 1.8. *The alternative classical relativistic electrodynamic model (68), which is intrinsically compatible with the classical Maxwell equations (14), allows the Hamiltonian formulation (66) with respect to the proper reference frame variables, where the Hamiltonian function is given by expression (67).*

The inference above is a natural candidate for experimental validation of our theory. It is strongly motivated by the following remark.

Remark 1.9. It is necessary to mention here that the Lorentz force expression (68) uses the particle momentum $p = mu$, where the dynamical "mass" $m := -\bar{W}$ satisfies condition (69). The latter gives rise to the following crucial relationship between the particle energy $\mathcal{E}_0$ and its rest mass $m_0 = -\bar{W}_0$ (for the velocity $u = 0$ at the initial time moment $t = 0$) :

$$\mathcal{E}_0 = m_0 \frac{1 - |\xi A_0/m_0|^2}{\sqrt{1 - 2|\xi A_0/m_0|^2}}, \tag{70}$$

or, equivalently, at the condition $|\xi A_0/m_0|^2 < \frac{1}{2}$

$$m_0 = \mathcal{E}_0 \sqrt{\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4|\xi A_0/\mathcal{E}_0|^2} + |\xi A_0/\mathcal{E}_0|^2}, \tag{71}$$

where $A_0 := A|_{t=0} \in \mathbb{E}^3$, which strongly differs from the classical expression $m_0 = \mathcal{E}_0 - \xi\varphi_0$, following from (37) and is not depending a priori on the external potential energy $\xi\varphi_0$. At $|\xi A_0/\mathcal{E}_0| \to 0$, the following asymptotical mass values follow from (71):

$$m_0 \simeq \mathcal{E}_0, \quad m_0^{(\pm)} \simeq \pm\sqrt{2}|\xi A_0|.$$

The first mass value $m_0 \simeq \mathcal{E}_0$ is physically correct, giving rise to the bounded charged particle energy $\mathcal{E}_0$, but the second mass values $m_0^{(\pm)} \simeq \pm\sqrt{2}|\xi A_0|$ are not physical, as they give rise to the vanishing denominator $\sqrt{1 - 2\left|\xi A_0/m_0^{(\pm)}\right|^2} \simeq 0$ in (70), being equivalent to the unboundedness of the charged particle energy modulus $|\mathcal{E}_0|$. It is also worth of mentioning that the sign of the mass m_0 always coincides with that of the energy $\mathcal{E}_0$.

To make this difference more clear, we now analyze the Lorentz force (63) from the Hamiltonian point of view based on the Lagrangian function (61). Thus, we obtain that the corresponding Hamiltonian function

$$
\begin{aligned}
H &:= \langle P, \dot{r}\rangle - \mathcal{L} = \langle P, \dot{r}\rangle + \bar{W}\sqrt{1+|\dot{r}|^2} - \xi\langle A, \dot{r}\rangle = \\
&= \langle P - \xi A, \dot{r}\rangle + \bar{W}\sqrt{1+|\dot{r}|^2} = \\
&= \frac{-\langle p, p\rangle}{\bar{W}\sqrt{1-|p|^2/\bar{W}^2}} + \frac{\bar{W}}{\sqrt{1-|p|^2/\bar{W}^2}} = \\
&= -\frac{\bar{W}^2 - |p|^2}{\sqrt{\bar{W}^2-|p|^2}} = -\sqrt{\bar{W}^2-|p|^2}.
\end{aligned}
\tag{72}
$$

Since $p = P - \xi A$, expression (72) assumes the final "*no interaction*" [50, 64, 49, 72] form

$$
H = -\sqrt{\bar{W}^2 - |P-\xi A|^2}, \tag{73}
$$

which is conserved with respect to the evolution equations (59) and (60), that is $dH/d\tau = dH/dt = 0$ for all $\tau, t \in \mathbb{R}$. These equations latter are equivalent to the following Hamiltonian system

$$
\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial P} = \frac{P - \xi A}{\sqrt{\bar{W}^2 - |P-\xi A|^2}}, \\
\dot{P} &= -\frac{\partial H}{\partial r} = \frac{\bar{W}\nabla\bar{W} - \nabla\langle \xi A, (P-\xi A)\rangle}{\sqrt{\bar{W}^2 - |P-\xi A|^2}},
\end{aligned}
\tag{74}
$$

as one can readily check by direct calculations. Actually, the first equation

$$
\begin{aligned}
\dot{r} = \frac{P-\xi A}{\sqrt{\bar{W}^2 - |P-\xi A|^2}} &= \frac{p}{\sqrt{\bar{W}^2-|p|^2}} = \\
&= \frac{mu}{\sqrt{\bar{W}^2-|p|^2}} = \frac{-\bar{W}u}{\sqrt{\bar{W}^2-|p|^2}} = \frac{u}{\sqrt{1-|u|^2}},
\end{aligned}
$$

holds, owing to the condition $d\tau = dt\sqrt{1-|u|^2}$ and definitions $p := mu$, $m = -\bar{W}$, postulated from the very beginning. Similarly we obtain that

$$
\dot{P} = -\frac{\nabla\bar{W}}{\sqrt{1-|p|^2/\bar{W}^2}} + \frac{\nabla\langle\xi A, u\rangle}{\sqrt{1-|p|^2/\bar{W}^2}} = -\frac{\nabla\bar{W}}{\sqrt{1-|u|^2}} + \frac{\nabla\langle\xi A, u\rangle}{\sqrt{1-|u|^2}},
$$

coincides with equation (62) in the evolution parameter $t \in \mathbb{R}$. This can be formulated as the next result.

Proposition 1.10. *The dual to the classical relativistic electrodynamic model (63) allows the canonical Hamiltonian formulation (74) with respect to the proper reference frame variables, where the Hamiltonian function is given by expression (73). Moreover, this formulation circumvents the "mass-potential energy" controversy attached to the classical electrodynamical model (36).*

The modified Lorentz force expression (63) and the related rest energy relationship are characterized by the following remark.

Remark 1.11. If we make use of the modified relativistic Lorentz force expression (63) as an alternative to the classical one of (38), the corresponding charged particle ξ energy expression (73) also gives rise to a true physically reasonable energy expression (at the velocity $u := 0 \in \mathbb{E}^3$ at the initial time moment $t = 0$); namely, $\mathcal{E}_0 = m_0$ instead of the physically controversial classical expression $\mathcal{E}_0 = m_0 + \xi\varphi_0$, where $\varphi_0 := \varphi|_{t=0}$, corresponding to the case (37).

1.6. The quantization of electrodynamics models in the vacuum field theory geometry-free approach

1.6.1. The problem setting

Recently [68, 69], we elaborated a new regular geometry-free approach to deriving the electrodynamics of a moving charged point particle ξ in an external electromagnetic field from first principles. This approach has, in part, reconciled the mass-energy controversy [17] in classical relativistic electrodynamics. Using the vacuum field theory approach initially proposed in [68, 69, 74], we reanalyzed this problem above from both the Lagrangian and Hamiltonian perspective and derived key expressions for the corresponding energy functions and Lorentz type forces acting on a moving charged point particle ξ.

Since all of our electrodynamics models were represented here in canonical Hamiltonian form, they are well suited to the application of Dirac quantization [14, 22] and the corresponding derivation of related Schrödinger type evolution equations. We describe these procedures in this section.

1.6.2. Free point particle electrodynamics model and its quantization

The charged point particle electrodynamics models, discussed in detail in Sections 2 and 3, were also considered in [69] from the dynamical point of view, where a Dirac quantization of the corresponding conserved energy expressions was attempted. However, from the canonical point of view, the true quantization procedure should be based on the relevant canonical Hamiltonian formulation of the models given in (65), (66) and (74).

In particular, consider a free charged point particle electrodynamics model characterized by (65) and having the Hamiltonian equations

$$\frac{dr}{d\tau} := \frac{\partial H}{\partial p} = -\frac{p}{\sqrt{\bar{W}^2 - |p|^2}}, \qquad \frac{dp}{d\tau} := -\frac{\partial H}{\partial r} = -\frac{\bar{W}\nabla\bar{W}}{\sqrt{\bar{W}^2 - |p|^2}},$$

where the function $\bar{W}: M^4 \to \mathbb{R}$, defined in the preceding sections, is the corresponding vacuum field potential characterizing medium field structure, $(r, p) \in$

$T^*(\mathbb{R}^3) \simeq \mathbb{E}^3 \times \mathbb{E}^3$ are the standard canonical coordinate-momentum variables on the cotangent space $T^*(\mathbb{R}^3)$, $\tau \in \mathbb{R}$ is the proper reference frame $\mathcal{K}_r$ time parameter of the moving particle, and $H : T^*(\mathbb{R}^3) \to \mathbb{R}$ is the Hamiltonian function

$$H := -\sqrt{\bar{W}^2 - |p|^2}, \tag{75}$$

expressed here and hereafter in light speed units. The proper reference frame $\mathcal{K}_r$, parameterized by variables $(\tau, r) \in \mathbb{E}^4$, is related to any other reference frame $\mathcal{K}$ in which our charged point particle ξ moves with velocity vector $u \in \mathbb{E}^3$. The frame $\mathcal{K}$ is parameterized by variables $(t, r) \in M^4$ via the Euclidean infinitesimal relationship $dt^2 = d\tau^2 + |dr|^2$, which is equivalent to the Minkowskian infinitesimal relationship $d\tau^2 = dt^2 - |dr|^2$. The Hamiltonian function (75) clearly satisfies the energy conservation conditions $dH/d\tau = dH/dt = 0$ for all $t, \tau \in \mathbb{R}$. This means that the suitable energy

$$\mathcal{E} = \sqrt{\bar{W}^2 - |p|^2} \tag{76}$$

can be treated by means of the Dirac quantization scheme [22, 23] to obtain, as $\hbar \to 0$, (or the light speed $c \to \infty$) the governing Schrödinger type dynamical equation. To do this following the approach in [68, 69], we need to make canonical operator replacements $\mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial \tau}$, $p \to \hat{p} := \frac{\hbar}{i}\nabla$, as $\hbar \to 0$, in the following energy expression:

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi) = (\psi, \hat{H}^+\hat{H}\psi), \tag{77}$$

where $(\cdot, \cdot)$ is the standard L_2-inner product. It follows from (76) that

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - |p|^2 = \hat{H}^+\hat{H} \tag{78}$$

is a suitable operator factorization in the Hilbert space $\mathcal{H} := L_2(\mathbb{R}^3; \mathbb{C})$ and $\psi \in \mathcal{H}$ is the corresponding normalized quantum vector state. Since the following elementary identity

$$\bar{W}^2 - |p|^2 = \bar{W}(1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}(1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}\bar{W} \tag{79}$$

holds, we can use (78) and (79) to define the operator

$$\hat{H} := (1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}\bar{W}. \tag{80}$$

Having calculated the operator expression (80) as $\hbar \to 0$ up to operator accuracy $O(\hbar^4)$, it is easy see that

$$\hat{H} = \frac{|p|^2}{2m(u)} + \bar{W} := -\frac{\hbar^2}{2m(u)}\nabla^2 + \bar{W}, \tag{81}$$

where we have taken into account the dynamical mass definition $m(u) := -\bar{W}$ (in the light speed units). Consequently, using (77) and (81), we obtain up to operator accuracy $O(\hbar^4)$ the following Schrödinger type evolution equation

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = -\frac{\hbar^2}{2m(u)}\nabla^2\psi + \bar{W}\psi \tag{82}$$

with respect to the rest reference frame $\mathcal{K}_r$ evolution parameter $\tau \in \mathbb{R}$. For a related evolution parameter $t \in \mathbb{R}$ parameterizing a reference frame $\mathcal{K}$, the equation (82) takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2 m_0}{2m(u)^2}\nabla^2\psi - m_0\psi. \tag{83}$$

Here we used the fact that it follows from (76) that the classical mass relationship $m(u) = m_0/\sqrt{1-|u|^2}$ holds, where $m_0 \in \mathbb{R}_+$ is the corresponding rest mass of our point particle ξ.

The linear Schrödinger equation (83) for the case $\hbar/c \to 0$ actually coincides with the well-known expression [22, 30, 50] from classical quantum mechanics.

1.6.3. Classical charged point particle electrodynamics model and its quantization

We start here from the first vacuum field theory reformulation of the classical charged point particle electrodynamics (introduced in Subsection 1.3.1) and based on the conserved Hamiltonian function (73)

$$H := -\sqrt{\bar{W}^2 - |P - \xi A|^2}, \tag{84}$$

where $\xi \in \mathbb{R}$ is the particle charge, $(\bar{W}, A) \in \mathbb{R} \times \mathbb{E}^3$ is the corresponding representation of the electromagnetic field potentials and $P \in \mathbb{E}^3$ is the common generalized particle-field momentum $P := p + \xi A$, $p := mu$, which satisfies the classical Lorentz force equation. Here $m := -\bar{W}$ is the observable dynamical mass of our charged particle, and $u \in \mathbb{E}^3$ is its velocity vector with respect to a chosen reference frame $\mathcal{K}$, all expressed in light speed units.

Our electrodynamics based on (84) is canonically Hamiltonian, so the Dirac quantization scheme

$$P \to \hat{P} := \frac{\hbar}{i}\nabla, \qquad \mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial\tau} \tag{85}$$

should be applied to the energy expression $\mathcal{E} := \sqrt{\bar{W}^2 - |P - \xi A|^2}$, following from the conservation conditions $dH/dt = 0 = dH/d\tau$, satisfied for all $\tau, t \in \mathbb{R}$.

Proceeding as above, we can factorize the operator $\hat{\mathcal{E}}^2$ as

$$\bar{W}^2 - |\hat{P} - \xi A|^2 =$$
$$= \bar{W}(1 - \bar{W}^{-1}|\hat{P} - \xi A|^2 \bar{W})^{1/2} (1 - \bar{W}^{-1}|\hat{P} - \xi A|^2 \bar{W}^{-1})^{1/2} \bar{W} := \hat{H}^+ \hat{H},$$

where (as $\hbar/c \to 0$, $\hbar c = \text{const}$) $\hat{H} := \frac{1}{2m(u)}\left|\frac{\hbar}{i}\nabla - \xi A\right|^2 + \bar{W}$ up to operator accuracy $O(\hbar^4)$. Hence, the related Schrödinger type evolution equation in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3;\mathbb{C})$ is

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = \frac{1}{2m(u)}|\tfrac{\hbar}{i}\nabla - \xi A|^2\psi + \bar{W}\psi \tag{86}$$

with respect to the proper reference frame $\mathcal{K}_r$ evolution parameter $\tau \in \mathbb{R}$, and corresponding Schrödinger type evolution equation with respect to the evolution parameter $t \in \mathbb{R}$ takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{m_0}{2m(u)^2}\left|\tfrac{\hbar}{i}\nabla - \xi A\right|^2\psi - m_0\psi.$$

The Schrödinger equation (86) (at $\hbar/c \to 0$) coincides [22, 51] with the classical quantum mechanics version.

1.6.4. Modified charged point particle electrodynamics model and its quantization

From the canonical viewpoint, we now turn to the true quantization procedure for the electrodynamics model, characterized by (53) and having the Hamiltonian function (67)

$$H := -\sqrt{\bar{W}^2 - \xi^2|A|^2 - |P|^2} - \frac{\xi\langle A, P\rangle}{\sqrt{\bar{W}^2 - \xi^2|A|^2 - |P|^2}}. \tag{87}$$

Accordingly the suitable energy function is

$$\mathcal{E} := \sqrt{\bar{W}^2 - \xi^2|A|^2 - |P|^2} + \frac{\xi\langle A, P\rangle}{\sqrt{\bar{W}^2 - \xi^2|A|^2 - |P|^2}}, \tag{88}$$

where, as before, $P := p + \xi A$, $p := mu$, $m := -\bar{W}$, is a conserved quantity for (53), which we will canonically quantize via the Dirac procedure (85). Toward this end, let us consider the quantum condition

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi), \quad (\psi, \psi) := 1, \tag{89}$$

where, by definition, $\hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial t}$ and $\psi \in \mathcal{H} = L_2(\mathbb{R}^3;\mathbb{C})$ is a respectively normalized quantum state vector. Making now use of the energy function (88), one readily computes that

$$\mathcal{E}^2 = \bar{W}^2 - |P - \xi A|^2 + \frac{\xi^2\langle A, P\rangle\langle P, A\rangle}{\bar{W}^2 - |P|^2},$$

which transforms by the canonical Dirac type quantization $P \to \hat{P} := \frac{\hbar}{i}\nabla$ into the symmetrized operator expression

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - |\hat{P} - \xi A|^2 + \frac{\xi^2 \langle A, \hat{P}\rangle\langle \hat{P}, A\rangle}{\bar{W}^2 - |\hat{P}|^2}. \tag{90}$$

Factorizing the operator (90) in the form $\hat{\mathcal{E}}^2 = \widehat{H}^+\widehat{H}$, and retaining only terms up to $O(\hbar^4)$ (as $\hbar/c \to 0$), we compute that

$$\hat{H} := \frac{1}{2m(u)}\left|\frac{\hbar}{i}\nabla - \xi A\right|^2 - \frac{\xi^2}{2m^3(u)}\langle A, \frac{\hbar}{i}\nabla\rangle\langle \frac{\hbar}{i}\nabla, A\rangle, \tag{91}$$

where, as before, $m(u) = -\bar{W}$ in light speed units. Thus, owing to (89) and (91), the resulting Schrödinger evolution equation is

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{H}\psi = \frac{1}{2m(u)}\left|\frac{\hbar}{i}\nabla - \xi A\right|^2\psi - \frac{\xi^2}{2m^3(u)}\langle A, \frac{\hbar}{i}\nabla\rangle\langle \frac{\hbar}{i}\nabla, A\rangle\psi \tag{92}$$

with respect to the proper reference frame proper evolution parameter $\tau \in \mathbb{R}$. The latter can be rewritten in an equivalent form as

$$i\hbar\frac{\partial\psi}{\partial\tau} = -\frac{\hbar^2}{2m(u)}\Delta\psi - \frac{1}{2m(u)}\langle[\frac{\hbar}{i}\nabla, \xi A]_+\rangle\psi - \frac{\xi^2}{2m^3(u)}\langle A, \frac{\hbar}{i}\nabla\rangle\langle \frac{\hbar}{i}\nabla, A\rangle\psi, \tag{93}$$

where $[\cdot,\cdot]_+$ means the formal anti-commutator of operators. Similarly one also obtains the related Schrödinger equation with respect to the time parameter $t \in \mathbb{R}$, which we shall not dwell upon here. The result (92) only slightly differs from the classical Schrödinger evolution equation (86). Simultaneously, its form (93) almost completely coincides with the classical ones from [22, 51, 64] modulo the evolution considered with respect to the proper reference time parameter $\tau \in \mathbb{R}$. This suggests that we must more thoroughly reexamine the physical motivation of the principles underlying the classical electrodynamic models, described by the Hamiltonian functions (84) and (87), giving rise to different Lorentz type force expressions. A more deep and extended analysis of this matter is forthcoming in a paper now in preparation.

2. Conclusions

All of dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding proper reference frames $\mathcal{K}_r$, parameterized by suitable time parameters $\tau \in \mathbb{R}$. Upon passing to the basic laboratory reference frame $\mathcal{K}$ with the time parameter $t \in \mathbb{R}$, the naturally related Hamiltonian structure is lost, giving rise to a new interpretation of the real particle motion. Namely, one that has an absolute sense only with respect to the proper rest reference system, and otherwise being completely relative with respect to all other reference frames. As for the Hamiltonian expressions (64), (67) and (73), one observes that they all depend strongly on the vacuum potential energy field function $\bar{W} : M^4 \to \mathbb{R}$, thereby avoiding the mass problem of the classical energy expression pointed out by L. Brillouin [17]. It should be noted that the canonical Dirac quantization procedure can be applied only to the corresponding dynamical field systems considered with respect to their proper reference frames.

Remark 2.1. *Some comments are in order concerning the classical relativity principle. We have obtained our results relying only on the natural notion of the proper reference frame and its suitable Lorentzian parametrization with respect to any other moving reference frames. It seems reasonable then that the true state changes of a moving charged particle ξ are exactly realized only with respect to its proper reference system. Then the only remaining question would be about the physical justification of the corresponding relationship between time parameters of moving and proper reference frames.*

The relationship between reference frames that we have used through is expressed as

$$d\tau = dt\sqrt{1 - |u|^2}, \tag{94}$$

where $u := dr/dt \in \mathbb{E}^3$ is the velocity vector with which the proper reference frame $\mathcal{K}_r$ moves with respect to another arbitrarily chosen reference frame $\mathcal{K}$. Expression (94) implies, in particular, that

$$dt^2 - |dr|^2 = d\tau^2, \tag{95}$$

which is identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [68, 69] from the governing equations of the vacuum potential field function $W : M^4 \to \mathbb{R}$ in the form

$$\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = \xi\rho, \quad \frac{\partial W}{\partial t} + \nabla(vW) = 0, \quad \frac{\partial \rho}{\partial t} + \nabla(v\rho) = 0, \tag{96}$$

which is *a priori* Lorentz invariant. Here $\rho \in \mathbb{R}$ is the charge density and $v := dr/dt$ the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (95) reflects this intrinsic structure of equations (96). Rewritten in a nonstandard Euclidean form $dt^2 = d\tau^2 + |dr|^2$, it gives rise to a completely different relationship between the reference frames $\mathcal{K}$ and $\mathcal{K}_r$, namely

$$dt = d\tau\sqrt{1 + |\dot{r}|^2},$$

where $\dot{r} := dr/d\tau$ is the related particle velocity with respect to the proper reference system. Thus, we observe that all our Lagrangian analysis in this Section is based on the corresponding functional expressions written in these "Euclidean" space-time coordinates and with respect to which the least action principle was applied. So we see that there are two alternatives: either to apply the least action principle to the corresponding Lagrangian functions expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference frame $\mathcal{K}$, or to apply the least action principle to the corresponding Lagrangian functions expressed in Euclidean space-time variables with respect to the proper reference frame $\mathcal{K}_r$.

This leads us to a slightly amusing but thought-provoking observation: according to our analysis, all results of classical special relativity related to the electrodynamics of charged point particles can be obtained (in a one-to-one correspondence) using our new definitions of the dynamical particle mass and the least action principle with respect to the associated Euclidean space-time variables in the proper reference system.

An additional remark concerning the quantization procedure of the proposed electrodynamics models is in order: if the dynamical vacuum field equations are expressed in canonical Hamiltonian form, as we have done in this paper, only straightforward technical details are required to quantize the equations and obtain the corresponding Schrödinger evolution equations in suitable Hilbert spaces of quantum states. There is another striking implication from our approach: the Einsteinian equivalence principle [30, 48, 50, 64] is rendered superfluous for our vacuum field theory of electromagnetism and gravity.

Using the canonical Hamiltonian formalism elaborated here for the alternative charged point particle electrodynamics models, we found it rather easy to treat the Dirac quantization. The results obtained compared favorably with classical quantization, but it must be admitted that we still have not given a compelling physical motivation for our new models. We plan to revisit this issue in future investigations. Another important aspect of our vacuum field theory geometry-free approach to combining the electrodynamics with the gravity, is the manner in which it singles out the decisive role of the rest reference frame $\mathcal{K}_r$. More precisely, all of our electrodynamics models allow both the Lagrangian and Hamiltonian formulations with respect to the proper reference system evolution parameter $\tau \in \mathbb{R}$, which are well suited the to canonical quantization. The physical nature of this fact remains is as yet not quite clear. In fact, as far as we know [48, 50, 52, 53, 64], there is no physically reasonable explanation of this decisive role of the rest reference system, except for that given by R. Feynman who argued in [30] that the relativistic expression for the classical Lorentz force (38) has physical sense only with respect to the proper reference frame variables $(\tau, r) \in \mathbb{R} \times \mathbb{E}^3$. In future research we plan to analyze the quantization scheme in more detail and begin work on formulating a vacuum quantum field theory of infinitely many particle systems.

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