

**EXISTENCE OF V-BOUNDED SOLUTIONS
FOR NONAUTONOMOUS NONLINEAR SYSTEMS
VIA THE WAŻEWSKI TOPOLOGICAL PRINCIPLE**

©2010 p. *Volodymyr LAGODA, Igor PARASYUK*

Kyiv National Taras Shevchenko Univesity
Volodymyrs'ka 64, Kyiv 01033
email: pio@univ.kiev.ua

Received August 30, 2010

We establish a number of new sufficient conditions for the existence of global (defined on the entire time axis) solutions of nonlinear nonautonomous systems by means of the Ważewski topological principle. The systems under consideration are characterized by the monotonicity property with respect to a certain auxiliary guiding function $W(t, x)$ depending on time and phase coordinates. Another auxiliary function $V(t, x)$, such that $\lim_{\|x\| \rightarrow \infty} V(t, x) = \infty$ for all $t \in \mathbb{R}$, is used to estimate the location of global solutions in the extended phase space. The approach developed is applied to Lagrangian systems and, in particular, to establish new sufficient conditions for the existence of almost periodic solutions.

1 Introduction

This paper is a modified and extended version of our e-print [1]. Its goal is to lay down some new sufficient conditions under which the nonlinear nonautonomous system of ODEs

$$\dot{x} = f(t, x), \tag{1}$$

УДК 517.9; MSC 34C11, 34C27, 70H12; guiding function, V-bounded solution, Ważewski topological principle, Lagrangian system, almost periodic solution

where $f : \Omega \mapsto \mathbb{R}^n$ ($\Omega \subseteq \mathbb{R}^{1+n}$), has a global solution $x(t)$ which exists on the entire time axis and possess the property that a given auxiliary spatially coercive function $V(t, x)$ (a time dependent norm surrogate) is bounded along the graph of $x(t)$. We especially focus on getting estimates for the function $V(t, x(t))$. The main results are obtained by using the Ważewski topological principle [2–5], and some of them generalize results of V. M. Cheresiz [6].

It should be noted that the Ważewski topological principle was successfully exploited for proving the existence of bounded solutions to some boundary value problems in [7] and to quasihomogeneous systems in [8, 9] (see also a discussion in [10]).

In order to apply the Ważewski principle, along with the function $V(\cdot)$ we use another auxiliary function $W(t, x)$ with positive derivative by virtue of the system (1) in the domain where $V > 0$. We call V and W *the estimating function* and *the guiding function* respectively and we say that together they form the V – W -pair of the system. Note that the term "guiding function" we borrow from [11] (originally – "guiding potential"). Basically topological method of guiding functions, which was developed by M. A. Krasnosel'ski and A. I. Perov, is an effective tool for proving the existence of bounded solutions of essentially nonlinear systems too (see the bibliography in [11, 12]). But, except [10, 14], in all papers known to us, only independent of time guiding functions were used.

In [6], the role of V – W -pair plays some function of Euclidean norm together with an indefinite nondegenerate quadratic form. It appears that in this case sufficient conditions for the existence of bounded solutions as well as the estimates of their norms coincide with those obtained by means of technique developed in [15, 16] for indefinitely monotone (not necessarily finite dimensional) systems.

We shall not mention here another interesting approaches in studying the existence problem of bounded solutions to nonlinear systems, because they have not been used in this paper. For the corresponding information the reader is referred to [17–32].

This paper is organized as follows. Section 2 contains necessary definitions, in particular, the notion of V – W -pair is introduced and some additional conditions imposed on estimating and guiding functions are described. In section 3, we prove two main theorems concerning the existence and the uniqueness of V -bounded solution to a nonlinear nonautonomous system possessing V – W -pair. In section 4 we show how the results of secti-

on 3 can be applied in the case where the estimating and guiding functions are constructed by means of nonautonomous quadratic forms. In this connection it should be pointed out that guiding quadratic forms play an important role in the theory of linear dichotomous systems with (integrally) bounded coefficients [33–35]. As an example of application of our technique, we generalize results of [17, 18] on the existence of bounded solutions to quasilinear nonautonomous system with exponentially dichotomic linear part. Finally, in section 5, the approach developed in section 3 is applied to a quasiconvex Lagrangian system of mechanical type with time-varying holonomic constraint. For such systems, we establish sufficient conditions for the existence of global solutions along which the Lagrangian function remains bounded. The case of almost periodic Lagrangian is also discussed. As an example we consider motion of a particle on helicoid under the impact of force of gravity and repelling potential field of force. Note that bounded and almost periodic solutions of globally strongly convex and Lipschitzian Lagrangian systems were studied in [26].

2 The definition of V–W-pair and the main assumptions

Let Ω be a domain in $\mathbb{R}^{1+n} = \{t \in \mathbb{R}\} \times \{x \in \mathbb{R}^n\}$ such that the projection of Ω on the time axis $\{t \in \mathbb{R}\}$ covers all this axis, and let $f(\cdot) \in C(\Omega \mapsto \mathbb{R}^n)$. It will be always assumed that each solution of the system (1) has the uniqueness property.

Definition 1. A function $V(\cdot) \in C(\mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R})$ of variables $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ will be called spatially coercive, if for any $t \in \mathbb{R}$ the function $V_t(\cdot) := V(t, \cdot) : \mathbb{R}^n \mapsto \mathbb{R}$ has the following properties: the level set $V_t^{-1}(0) := \{x \in \mathbb{R}^n : V_t(x) = 0\}$ is nonempty and $\lim_{\|x\| \rightarrow \infty} V_t(x) = \infty$. If in addition $V(\cdot) \in C^1(\mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R})$ and $\|\frac{\partial V_t(x)}{\partial x}\| > 0$ once $V_t(x) > 0$, then $V(\cdot)$ will be called a regular spatially coercive function.

Note that for each $t \in \mathbb{R}$ and each $v > 0$ the level set $V_t^{-1}(v)$ of regular spatially coercive function $V(\cdot)$ is a compact connected and simply connected hypersurface which, thus, is homeomorphic to $(n - 1)$ -dimensional sphere; in addition, if $v_2 > v_1$, then the set $V_t^{-1}((-\infty, v_1]) := \{x \in \mathbb{R}^n : V_t(x) \leq v_1\}$ is a proper subset of the set $V_t^{-1}((-\infty, v_2])$.

Definition 2. For a spatially coercive function $V(\cdot)$, a global solution $x(t)$,

$t \in I$, of the system (1) is said to be V -bounded if

$$\sup_{t \in I} V(t, x(t)) < \infty,$$

and $V(\cdot)$ is then called an estimating function.

For any $U(\cdot) \in C^1(\Omega \mapsto \mathbb{R})$, define

$$\dot{U}_f := \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} f.$$

Definition 3. A function $W(\cdot) \in C^1(\Omega \mapsto \mathbb{R})$ will be called a guiding function concordant with a spatially coercive function $V(\cdot)$ if $\Omega \cap V^{-1}((0, \infty)) \neq \emptyset$ and $\dot{W}_f(t, x) > 0$ for all $(t, x) \in \Omega \cap V^{-1}((0, \infty))$.

Definition 4. A regular spatially coercive function $V(\cdot)$ together with a concordant guiding function $W(\cdot)$ will be called a V - W -pair of the system (1).

Denote by $\Pi_t := \{t\} \times \mathbb{R}^n$ the "vertical" hyperplane in $\mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n$ passing through $(t, 0)$ and for any set $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}^n$ denote by \mathcal{A}_t the natural projection of the set $\Pi_t \cap \mathcal{A}$ onto \mathbb{R}^n .

In so far, we suppose that the system (1) has a V - W -pair which satisfies the following additional conditions:

- (A): there exist numbers w^* , w_* ($w^* > w_*$), $c^* > 0$, $c_* \in [0, \infty]$, and a connected component \mathcal{W} of the set $W^{-1}((w_*, w^*))$ such that for any $t \in \mathbb{R}$ the number w^* belongs to the range of function $W_t(\cdot) := W(t, \cdot) : \Omega_t \mapsto \mathbb{R}$, the set $V^{-1}((-\infty, 0])$ belongs to \mathcal{W} , and the following inequalities hold

$$-c_* \dot{W}_f(t, x) \leq \dot{V}_f(t, x) \leq c^* \dot{W}_f(t, x) \quad \forall (t, x) \in V^{-1}([0, \infty)) \cap \mathcal{W}. \quad (2)$$

Note that from condition (A), it follows that

$$\begin{aligned} w_0(t) &:= \min\{W_t(x) : x \in V_t^{-1}(0)\} > w_*, \\ w^0(t) &:= \max\{W_t(x) : x \in V_t^{-1}(0)\} < w^*, \end{aligned} \quad (3)$$

thus, the set $\partial\mathcal{W} \cap W^{-1}(w^*)$ coincides with the set of exit points from \mathcal{W} , each point of $\partial\mathcal{W} \cap W^{-1}(w^*)$ being a strict exit point. Denote

$$\mathcal{W}^{se} := \partial\mathcal{W} \cap W^{-1}(w^*).$$

(B): the function

$$\alpha(t) := \inf \left\{ \dot{W}_f(t, x) : x \in V_t^{-1}((0, \infty)) \cap \mathcal{W}_t \right\}$$

has the property $\int_{-\infty}^0 \alpha(s) ds = \int_0^{\infty} \alpha(s) ds = \infty$.

(C): for any sufficiently large by absolute value negative t , the Ważewski condition is fulfilled: there exists a bounded subset \mathcal{M}_t of the set $\mathcal{W}_t \cup \mathcal{W}_t^{se}$ such that the set $\{t\} \times (\mathcal{M}_t \cap \mathcal{W}_t^{se})$ is a retract of $\{(s, x) \in \mathcal{W}^{se} : s \geq t\}$, but is not a retract of $\{t\} \times \mathcal{M}_t$.

Remark 1. In the case where $V(\cdot)$ and $W(\cdot)$ do not depend of t , one can consider the inequalities (2) as an analogue of the regularity condition for the guiding function $W(\cdot)$ (see [11]). The main consequence of regularity in this case is that the pair $cW(\cdot)$, $cW(\cdot) - V(\cdot)$ (or $cW(\cdot)$, $cW(\cdot) + V$) is a complete set of guiding functions for any $c > c^*$ (for any $c > c_*$ if $c_* < \infty$).

Remark 2. The condition (C) is fulfilled if for any negative sufficiently large by absolute value t there exists a compact manifold \mathcal{M}_t with border $\partial\mathcal{M}_t$ such that the interior of \mathcal{M}_t belongs to \mathcal{W}_t and the set $\{t\} \times (\mathcal{M}_t \cap \mathcal{W}_t^{se})$ is a retract of $\{(s, x) \in \mathcal{W}^{se} : s \geq t\}$. In fact, as is well known, $\partial\mathcal{M}_t$ is not a retract of \mathcal{M}_t .

Taking into account that \mathcal{W}^{se} is a union of connected components of regular level hypersurface $W^{-1}(w^*)$, the condition (C) can be replaced by the following weaker condition:

(C'): there exists a bounded subset \mathcal{M}_t of the set $\mathcal{W}_t \cup \mathcal{W}_t^{se}$ which cannot be continuously imbedded into the set $\{(s, x) \in \mathcal{W}^{se} : s \geq t\}$ in such a way that the image of $\mathcal{M}_t \cap \mathcal{W}_t^{se}$ is $\{t\} \times (\mathcal{M}_t \cap \mathcal{W}_t^{se})$.

3 The existence and the uniqueness of V-bounded solution

The lemma given below open the door to estimation of solutions of the system (1) by means of V–W-pair.

Lemma 1. *Suppose that the system (1) has V–W-pair satisfying the condition (A). Let $x(t)$ be such a solution of (1) that $(t, x(t)) \in \mathcal{W}$ for all $t \in [t_0, t_1]$.*

Then the following assertions are true:

1. if $V(t, x(t)) > 0$ for all $t \in (t_0, t_1]$, then

$$V(t, x(t)) \leq V(t_0, x(t_0)) + c^* [w^* - W(t_0, x(t_0))] \quad \forall t \in [t_0, t_1]; \quad (4)$$

2. if $V(t, x(t)) > 0$ for all $t \in (t_0, t_1)$ and $V(t_1, x(t_1)) = 0$, then

$$V(t, x(t)) \leq \frac{c_*}{c_* + c^*} V(t_0, x(t_0)) + \frac{c_* c^*}{c_* + c^*} [w^0(t_1) - W(t_0, x(t_0))] \quad (5)$$

$$\forall t \in [t_0, t_1];$$

3. if the condition (B) is fulfilled, $V(t_0, x(t_0)) \geq 0$ and

$$\int_{t_0}^{t_1} \alpha(s) ds \geq w^* - w_*, \quad (6)$$

then there exists $\tau \in (t_0, t_1)$ such that $V(\tau, x(\tau)) = 0$.

Proof. Let the condition (A) is fulfilled. Put $v(t) := V(t, x(t))$, $w(t) = W(t, x(t))$. The inequality (4) obviously follows from $\dot{v}(t) \leq c^* \dot{w}(t)$, $t \in [t_0, t_1]$. In order to prove the inequality (5), denote by \hat{t} any point where $v(t)$ reaches its maximum on $[t_0, t_1]$ and observe that

$$\begin{aligned} w(t_1) - w(t_0) &= \int_{t_0}^{\hat{t}} \dot{w}(t) dt + \int_{\hat{t}}^{t_1} \dot{w}(t) dt \geq \frac{1}{c^*} \int_{t_0}^{\hat{t}} \dot{v}(t) dt - \frac{1}{c_*} \int_{\hat{t}}^{t_1} \dot{v}(t) dt = \\ &= \frac{v(\hat{t}) - v(t_0)}{c^*} + \frac{v(\hat{t})}{c_*} \geq \frac{(c_* + c^*)v(t)}{c_* c^*} - \frac{v(t_0)}{c_*}. \end{aligned}$$

Since $v(t_1) = 0$, then $w(t_1) \leq w^0(t_1)$ and we get (5).

Now let the condition (B) is fulfilled and $v(t_0) \geq 0$. If we assume that $v(t) > 0$ on (t_0, t_1) , then

$$\int_{t_0}^{t_1} \alpha(t) dt \leq \int_{t_0}^{t_1} \dot{w}(t) dt = w(t_1) - w(t_0) < w^* - w_*.$$

This contradicts the inequality (6). □

Put

$$\omega_0 := \inf_{t \in \mathbb{R}} w_0(t), \quad \omega^0 := \sup_{t \in \mathbb{R}} w^0(t), \quad (7)$$

$$\nu := \liminf_{t \rightarrow -\infty} \sup \{V_t(x) - c^* W_t(x) : x \in \mathcal{M}_t \cap V_t^{-1}((0, \infty))\}. \quad (8)$$

Now we are in position to prove the following theorem.

Theorem 1. *Assume that the system (1) has a V-W-pair satisfying the conditions (A),(B),(C) (or (C')), and $\nu < \infty$. Let there exists a number $V^* > c^*w^* + \max \{\nu, -c^*\omega_0\}$ such that*

$$\text{cls} (V^{-1} ([0, V^*]) \cap \mathcal{W}) \subset \Omega$$

(here cls means the closure operation). Then the system (1) has a V-bounded global solution $x_*(t)$, $t \in \mathbb{R}$, which for all $t \in \mathbb{R}$ satisfies the inequalities

$$\begin{aligned} V(t, x_*(t)) &\leq \frac{c_*c^*}{c_* + c^*} \left[\sup_{t \leq s \leq \tau_+(t)} w^0(s) - \inf_{\tau_-(t) \leq s \leq t} w_0(s) \right] \leq & (9) \\ &\leq \frac{c_*c^*}{c_* + c^*} (\omega^0 - \omega_0), \\ \omega_0 \leq W(t, x_*(t)) &\leq \omega^0 & (10) \end{aligned}$$

where $\tau_+(t)$ and $\tau_-(t)$ are, respectively, the roots of equations

$$\int_t^{\tau_+} \alpha(s) ds = \omega^0 - \omega_0, \quad \int_{\tau_-}^t \alpha(s) ds = \omega^0 - \omega_0.$$

Proof. From definitions of ν and V^* it follows that there exists a sequence $t_j \rightarrow -\infty, j \rightarrow \infty$, such that

$$c^*w^* + \sup \left\{ V_{t_j}(x) - c^*W_{t_j}(x) : x \in \mathcal{M}_{t_j} \cap V_{t_j}^{-1} ((0, \infty)) \right\} < V^*. \quad (11)$$

In view of condition (C) (or (C')) from Ważewski principle it follows that for any j there exists a point $x_{0j} \in \mathcal{M}_{t_j}$ such that the global solution $x_j(t)$, $t \in I_j$, which satisfies the initial condition $x_j(t_j) = x_{0j}$ has the property

$$(t, x_j(t)) \in \mathcal{W} \quad \forall t \in [t_j, \infty) \cap I_j.$$

Let us show that

$$v_j(t) := V(t, x_j(t)) < V^* \quad \forall t \in I_j \cap [t_j, \infty). \quad (12)$$

For any natural j , it is sufficient to consider the following cases: (I) $v_j(t) > 0$ for all $t \in I_j \cap (t_j, \infty)$; (II) $v_j(t_j) \geq 0$, there exist $t_* \geq t_j$ and $t^* \geq t_*$ such that $v(t_*) = v(t^*) = 0$, $v(t) > 0$ for all $t \in I_j \cap (t^*, \infty)$, and if $t_* > t_j$, then $v_j(t) > 0$ for all $t \in (t_j, t_*)$; (III) there exist increasing sequences t_{k*}, t_k^* in

$I_j \cap [t_j, \infty)$, $k \in \mathbb{N}$, such that $t_{k*} < t_k^*$, $t_{k+1,*} \geq t_k^*$, $t_k^* \rightarrow \sup \{t \in I_j\}$, $k \rightarrow \infty$, and

$$v_j(t_{k*}) = v_j(t_k^*) = 0, \\ v_j(t) > 0 \quad \forall t \in (t_{k*}, t_k^*), \quad v_j(t) \leq 0 \quad \forall t \in (I_j \setminus \bigcup_{k=1}^{\infty} (t_{k*}, t_k^*) \cap [t_j, \infty).$$

In the case (I), observe that for sufficiently small $\delta > 0$ we have

$$v_j(t_j) + c^* [w^* - W(t_j, x_{0j})] < c^* w^* + \nu + \delta < V^*.$$

Now the inequality (12) immediately follows from (4).

In the case (II), observe that

$$v(t^*) + c^* [w^* - W(t^*, x_j(t^*))] \leq c^* [w^* - w_0(t^*)] \leq c^* [w^* - \omega_0] < V^*.$$

Thus, similarly to the case (I), we obtain the estimate $v_j(t) < V^*$ for all $t \in I_j \cap [t^*, \infty)$. Next, if $t_* > t_j$, then $W(t_j, x_j(t_j)) \leq W(t_*, x_j(t_*)) \leq w^0(t_*)$ and from (5) it follows that

$$v_j(t) \leq \frac{c_*}{c_* + c^*} v_j(t_j) + \frac{c_* c^*}{c_* + c^*} [w^0(t_*) - W(t_j, x_j(t_j))] < \frac{c_*}{c_* + c^*} V^* \leq V^* \\ \forall t \in [t_j, t_*].$$

If now $t_* = t^*$, then the inequality (12) holds true. And if $t_* < t^*$, then for any successive zeroes $t_1, t_2 \in [t_*, t^*]$ of function $v_j(t)$ from (5) it follows that

$$v_j(t) \leq \frac{c_* c^*}{c_* + c^*} [w^0(t_2) - w_0(t_1)] < \frac{c_* c^*}{c_* + c^*} [\omega^0 - \omega_0] < V^* \quad \forall t \in [t_1, t_2].$$

Thus, we obtain inequality (12) in the case (II), and now it becomes obvious that this inequality is valid also for the case (III).

The above reasoning allows us to make conclusion that in view of definition of V^* the graph of $x_j(t)$, $t \in I_j \cap [t_j, \infty)$, is contained in a closed subset of \mathcal{W} . This yields inclusion $[t_j, \infty) \subset I_j$.

Now we are in position to prove the existence of V -bounded solution $x_*(t)$ by the known scheme (see, e.g., [6, 9, 11]). Namely, if we denote by $x(t, t_0, x_0)$ the solution which for $t = t_0$ takes the value x_0 , then setting $\xi_j := x_j(0)$, we obtain the equalities

$$x_j(t) = x(t, 0, x_j(0)) = x(t, 0, \xi_j), \quad t \in [t_j, \infty).$$

Having selected from the sequence $\xi_j \in \text{cls}(V_0^{-1}([0, V^*]) \cap \mathcal{W}_0) \subset \Omega_0$ a subsequence converging to x_* , put $x_*(t) := x(t, 0, x_*)$. Using reductio ad absurdum reasoning it is easy to show that on the maximal existence interval I of this solution we have the inclusion $(t, x_*(t)) \in \text{cls}(V^{-1}([0, V_*]) \cap \mathcal{W})$. Therefore $I = \mathbb{R}$.

Now we are able to establish a sharper estimate for $v_*(t) := V(t, x_*(t))$. Namely, for any $t \in \mathbb{R}$ such that $v_*(t) > 0$, in virtue of Lemma 1, the point t lies between two successive zeroes $t_*(t), t^*(t)$ of $v_*(t)$ each of which is contained in the segment $[\tau_-(t), \tau_+(t)]$. Then the inequality (9) easily follows from (5) once we put there $t_0 = t_*(t), t_1 = t^*(t)$. \square

The following theorem establishes sufficient conditions for the uniqueness of V-bounded solution.

Theorem 2. *Let $\tilde{\Omega}$ be a subset of the domain Ω and let*

$$\tilde{\Omega}^* := \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^{2n} : (t, x) \in \tilde{\Omega}, (t, y) \in \tilde{\Omega}\}.$$

Suppose that there exist functions $V(\cdot) : \mathbb{C}^1(\mathbb{R}^{1+n}) \mapsto \mathbb{R}$, $U(\cdot) \in \mathbb{C}^1(\tilde{\Omega}^ \mapsto \mathbb{R})$, $\eta(\cdot) \in \mathbb{C}(\mathbb{R}_+ \mapsto \mathbb{R}_+)$, and $\beta(\cdot) \in \mathbb{C}(\mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+)$ such that:*

1) *the function $V(\cdot)$ is spatially coercive and the function $\eta(\cdot)$ is positive definite;*

2) *the function*

$$\dot{U}_{(f,f)}(t, x, y) := \frac{\partial U(t,x,y)}{\partial t} + \frac{\partial U(t,x,y)}{\partial x} f(t, x) + \frac{\partial U(t,x,y)}{\partial y} f(t, y)$$

satisfies the inequality

$$\dot{U}_{(f,f)}(t, x, y) \geq \beta(t, r)\eta(|U(t, x, y)|) \quad \forall (x, y) \in \tilde{V}_t^{-1}((-\infty, r]) \cap \tilde{\Omega}_t^*,$$

with $\tilde{V}(t, x, y) := \max\{V(t, x), V(t, y)\}$, and takes positive value at any point $(t, x, y) \in \tilde{\Omega}^$ such that $x \neq y$ and $U(t, x, y) = 0$ (if the set of such points is nonempty);*

3) *for any sufficiently large $r \geq 0$, the functions $\beta(\cdot), h(u) := \int_1^u \frac{ds}{\eta(s)}$ ($u > 0$), and*

$$b(t, r) := \max \left\{ |U(t, x, y)| : (x, y) \in \tilde{V}_t^{-1}((-\infty, r]) \cap \tilde{\Omega}_t^* \right\}$$

satisfy the conditions

$$\int_0^{\pm\infty} \beta(s, r) ds = \pm\infty, \quad \liminf_{t \rightarrow \pm\infty} \frac{h(b(t, r))}{\left| \int_0^t \beta(s, r) ds \right|} < 1.$$

Then the system (1) cannot have two different V-bounded solutions $x(t), y(t), t \in \mathbb{R}$, whose graphs lie in $\tilde{\Omega}$.

Proof. Suppose that the system (1) has a pair of solutions $x(t), y(t), t \in \mathbb{R}$, such that $(t, x(t)), (t, y(t)) \in \tilde{\Omega}$ and $x(t) \neq y(t)$ for all $t \in \mathbb{R}$. Let us show that at least one of these solutions is not V-bounded.

Using reductio ad absurdum reasoning we suppose that there exists sufficiently large $r > 0$ such that $|\tilde{V}(t, x(t), y(t))| \leq r$ for all $t \in \mathbb{R}$. Consider the function $u(t) := U(t, x(t), y(t))$. By condition, the function $u(\cdot)$ is nondecreasing. Hence, there exist limits $u_* = \lim_{t \rightarrow -\infty} u(t), u^* = \lim_{t \rightarrow \infty} u(t)$ (either finite or infinite).

Firstly, suppose that $u_* \geq 0$. If $u(0) = 0$, then by condition 2) $\dot{u}(0) > 0$. Hence, in this case, as well as in the case where $u(0) > 0$, we have the inequality $u(t) > 0$ for all $t > 0$. Now the condition 2) yields

$$h(u(t)) - h(u(t_0)) \geq \int_{t_0}^0 \beta(s, r) ds + \int_0^t \beta(s, r) ds \quad \forall t_0 > 0, \forall t \geq t_0.$$

This implies that

$$h(b(t, r)) - h(u(0)) + \int_0^{t_0} \beta(s, r) ds \geq \int_0^t \beta(s, r) ds \quad \forall t \geq t_0,$$

and we arrive at contradiction with assumption 3).

Now suppose that $u_* < 0$. Then there exists t' such that $u(t') < 0$. Thus, $u(t) \leq u(t')$ for all $t < t'$. Then

$$\begin{aligned} \int_{u(t)}^{u(t')} \frac{ds}{\eta(-s)} &\geq \int_t^{t'} \beta(s, r) ds \quad \Rightarrow \quad h(|u(t)|) - h(|u(t')|) \geq \\ &\geq \left| \int_0^t \beta(s, r) ds \right| + \int_0^{t'} \beta(s, r) ds \end{aligned}$$

from whence, as above, we again arrive at contradiction. □

Remark 3. If $\int_0^1 \frac{1}{\eta(u)} du < \infty$, then the condition 3) can be replaced by the following one:

$$\liminf_{t \rightarrow \infty} \frac{h(b(t, r)) + h(b(-t, r)) - 2h(0)}{\int_{-t}^t \beta(s, r) ds} < 1.$$

4 Studying V-bounded solutions by means of quadratic forms

Denote by $\langle \cdot, \cdot \rangle$ a scalar product in \mathbb{R}^n , and let $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$. In this section, the case will be considered where the guiding function is a time dependent nondegenerate indefinite quadratic form $\langle S(t)x, x \rangle$. In more detail, the mapping $S(\cdot) \in C^1(\mathbb{R} \mapsto \text{Aut}(\mathbb{R}^n))$ assumed to have the following property:

- (a): for any $t \in \mathbb{R}$ the operator $S(t)$ is symmetric and there exists a decomposition of \mathbb{R}^n into direct sum of two $S(t)$ -invariant subspaces $\mathbb{L}_+(t), \mathbb{L}_-(t)$ such that the restriction of $S(t)$ on $\mathbb{L}_+(t)$ (on $\mathbb{L}_-(t)$) is a positive definite (negative definite) operator.

Observe that since the subspaces $\mathbb{L}_+(t), \mathbb{L}_-(t)$ are mutually orthogonal, the corresponding projectors $P_{\pm}(t) : \mathbb{R}^n \mapsto \mathbb{L}_{\pm}(t)$ are symmetric.

It appears that the function $W(t, x) = \langle S(t)x, x \rangle$ generates a set \mathcal{W} possessing the Ważewski property (C). For the sake of completeness we give here a proof of the corresponding statement.

Lemma 2. *Let $W(t, x) := \langle S(t)x, x \rangle$ and let $S(\cdot)$ has the property (a). Then for any $w > 0$, $t_0 \in \mathbb{R}$ there exists a retraction of the set $W^{-1}(w)$ to the ellipsoid $\{t_0\} \times (W_{t_0}^{-1}(w) \cap \mathbb{L}_+(t_0))$.*

Proof. From $S(t)$ -invariance of subspaces $\mathbb{L}_+(t), \mathbb{L}_-(t)$ it follows that $P_{\pm}(t)S(t) = S(t)P_{\pm}(t)$ and, as a consequence, we have the representation

$$\begin{aligned} S(t) &= (P_+(t) + P_-(t))S(t)(P_+(t) + P_-(t)) = \\ &= P_+(t)S(t)P_+(t) + P_-(t)S(t)P_-(t). \end{aligned}$$

Put

$$S_+(t) := P_+(t)S(t)P_+(t), \quad S_-(t) := P_-(t)S(t)P_-(t).$$

Obviously, the kernel of the operator $S_+(t)$ (operator $S_-(t)$) is the subspace $\mathbb{L}_-(t)$ (subspace $\mathbb{L}_+(t)$), and the restriction of this operator on $\mathbb{L}_+(t)$ (on $\mathbb{L}_-(t)$) is a positively definite (negatively definite) operator.

Now observe that for arbitrary $t \in \mathbb{R}$ and $w > 0$ there exists a retraction of $W_t^{-1}(w) = \{x \in \mathbb{R}^n : \langle S(t)x, x \rangle = w\}$ to the intersection of this set with the subspace $\mathbb{L}_+(t)$. In fact, one can define such a retraction by a mapping $x \mapsto \varpi(t, x)P_+(t)x$, provided that the scalar function $\varpi(t, x)$ is determined

from condition $\langle S_+(t)\varpi(t,x)x, \varpi(t,x)x \rangle = w$ for all $x \in W_t^{-1}(w)$. Since $w > 0$, then $W_t^{-1}(w) \cap \mathbb{L}_-(t) = \emptyset$, and hence, $\langle S_+(t)x, x \rangle > 0$ for all $x \in W_t^{-1}(w)$. Therefore

$$\varpi(t, x) = \sqrt{\frac{w}{\langle S_+(t)x, x \rangle}}.$$

Now it remains only to show that the set $\{t_0\} \times W_t^{-1}(w) = W^{-1}(w) \cap \Pi_{t_0}$ is a retract of $W^{-1}(w)$. Introduce the operator $R(t) := \sqrt{S^2(t)} = S_+(t) - S_-(t)$. Then we get

$$S(t) = R(t)(P_+(t) - P_-(t)) = (P_+(t) - P_-(t))R(t).$$

The quadratic form $\langle S(t)x, x \rangle$ by means of the substitution $x = \left[\sqrt{R(t)} \right]^{-1} y$ is reduced to $\langle (P_+(t) - P_-(t))y, y \rangle$. Obviously, $P_+(t) - P_-(t)$ is a symmetric orthogonal inversion operator:

$$(P_+(t) - P_-(t))^* = P_+(t) - P_-(t), \quad (P_+(t) - P_-(t))^2 = E.$$

From the representation via the Riesz formula (see, e.g., [33, c. 34]) it follows that the projectors $P_{\pm}(t)$ smoothly depend on parameter. Therefore the mutually orthogonal subspaces $\mathbb{L}_+(t)$ and $\mathbb{L}_-(t)$ have constant dimensions n_+ , n_- and define smooth curves γ_+ , γ_- in Grassmannian manifolds $G(n, n_+)$ and $G(n, n_-)$ respectively. Since $G(n, n_+)$ is a base space of a principal fiber bundle, namely, $G(n, n_+) = O(n)/O(n_+) \times O(n_-)$, then there exists a smooth curve $Q(t)$ in $O(n)$, which is projected onto $\gamma_+(t)$, the operator $Q(t_0)$ being the identity element E of the group $O(n)$. Obviously, $\mathbb{L}_+(t) = Q(t)\mathbb{L}_+(t_0)$ and, as a consequence,

$$P_{\pm}(t) = Q(t)P_{\pm}(t_0)Q^{-1}(t).$$

From the above reasoning it follows that the change of variables

$$x = \left[\sqrt{R(t)} \right]^{-1} Q(t)\sqrt{R(t_0)}y$$

reduces the quadratic form $W(t, x) := \langle S(t)x, x \rangle$ to $W(t_0, y) = \langle S(t_0)y, y \rangle$, and then the mapping

$$\mathbb{R} \times \mathbb{R}^n \mapsto \{t_0\} \times \mathbb{R}^n : \quad (t, x) \mapsto \left(t_0, \sqrt{R(t)}Q^{-1}(t) \left[\sqrt{R(t_0)} \right]^{-1} x \right)$$

determines a retraction of the set $W^{-1}(w)$ to the set $W^{-1}(w) \cap \Pi_{t_0}$. □

Now consider the quasilinear system

$$\dot{x} = f(t, x) := A(t)x + g(t, x) \tag{13}$$

and assume that the following conditions hold:

- (b): the mapping $A(\cdot) \in C(\mathbb{R} \mapsto \text{Hom}(\mathbb{R}^n))$ is such that $\sup_{t \in \mathbb{R}} \|A(t)\| =: a < \infty$ and the linear system $\dot{x} = A(t)x$ is exponentially dichotomic on \mathbb{R} ; i.e. there exists a mapping $C(\cdot) \in C^1(\mathbb{R} \mapsto \text{Aut}(\mathbb{R}^n))$ possessing the property (a) with $S(t) = C(t)$, and, in addition,

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|C(t)\| =: c < \infty, \quad \inf_{t \in \mathbb{R}} |\det C(t)| =: \sigma > 0, \\ \langle (2C(t)A(t) + \dot{C}(t))x, x \rangle \geq \|x\|^2 \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^n \end{aligned}$$

(see, e.g. [34, 35]).

- (c): there exist $k > 0$ and $\varphi(\cdot) \in C^1(\mathbb{R} \mapsto (0, \infty))$ such that $\sup_{t \in \mathbb{R}} \frac{|\dot{\varphi}(t)|}{\varphi(t)} =: l < \infty$ and the mapping $g(\cdot) \in C(\mathbb{R}^{1+n} \mapsto \mathbb{R}^n)$ satisfies the inequality

$$\|g(t, x)\| \leq k \|x\| + \varphi(t) \quad \forall (t, x) \in \mathbb{R}^{1+n}.$$

The well known approach to establish sufficient conditions for the existence of bounded solutions to (13) is based on the method of integral equations which allows to apply different versions of fixed point theorems (see, e.g. [17, 18]). Our goal is to show that by means of V-W-pair one can not only establish the existence of bounded solutions (in the case where $\varphi(t)$ is bounded), but also show how their asymptotic behavior depends on $\varphi(t)$ as $t \rightarrow \pm\infty$.

For any $t \in \mathbb{R}$, put

$$\begin{aligned} \lambda_C^+(t) &:= \max_{\|x\|=1} \langle C(t)x, x \rangle, \quad \lambda_C^-(t) := \min_{\|x\|=1} \langle C(t)x, x \rangle, \\ \lambda_{C, \min}^+(t) &:= \min \{ \langle C(t)x, x \rangle : \|x\| = 1, x \in \mathbb{L}_+(t) \} \end{aligned}$$

and

$$F(r) := \begin{cases} \frac{d}{m} r^2 + 2 \left(\frac{c}{m} + \frac{d}{m^2} \right) \left(\frac{1}{m} \ln(mr + 1) - r \right) & \text{if } r \geq \frac{c}{d}, \\ F\left(\frac{c}{d}\right) & \text{if } 0 \leq r < \frac{c}{d}, \end{cases}$$

where $d := \frac{1}{2} - c(k + l)$, $m := a + k + l$.

Theorem 3. *Let the conditions (b),(c) hold true and let the numbers c, k, l satisfy the inequality*

$$c(k + l) < \frac{1}{2}.$$

Then the system (13) has a solution $x_(t)$, $t \in \mathbb{R}$, such that*

$$\|x_*(t)\| \leq r_*\varphi(t) \quad \forall t \in \mathbb{R}, \tag{14}$$

where r_ is the root of equation*

$$F(r) = F\left(\frac{c}{d}\right) + \frac{c^2}{2d^2} \sup_{t \in \mathbb{R}} \left[\sup_{s \geq t} \lambda_C^+(s) - \inf_{s \leq t} \lambda_C^-(s) \right].$$

If, in addition,

$$\|g(t, x) - g(t, y)\| \leq k\|x - y\| \quad \forall (t, x, y) \in \mathbb{R}^{1+2n},$$

then $x_(t)$ is a unique solution of the system (13) for which the ratio $\frac{\|x\|}{\varphi(t)}$ is bounded on \mathbb{R} .*

Proof. First, we show that the system (13) has the following V-W-pair

$$V(t, x) := F\left(\frac{\|x\|}{\varphi(t)}\right) - F(r_0), \quad W(t, x) = \frac{\langle C(t)x, x \rangle}{\varphi^2(t)}, \tag{15}$$

where r_0 is an arbitrary number greater than c/d . In fact, from the inequalities

$$\begin{aligned} \left| \frac{d}{dt} \left[\frac{\|x\|^2}{\varphi^2(t)} \right]_f \right| &\leq \frac{2}{\varphi^2(t)} \left[(a + k) \|x\|^2 + \varphi(t)\|x\| \right] + 2 \frac{|\dot{\varphi}(t)|}{\varphi^3(t)} \|x\|^2 \leq \\ &\leq 2m \frac{\|x\|^2}{\varphi^2(t)} + 2 \frac{\|x\|}{\varphi(t)}, \\ \frac{d}{dt} \left[\frac{\langle C(t)x, x \rangle}{\varphi^2(t)} \right]_f &\geq \frac{1}{\varphi^2(t)} \left[(1 - 2ck) \|x\|^2 - 2c\varphi(t)\|x\| \right] - 2c \frac{|\dot{\varphi}(t)|}{\varphi^3(t)} \|x\|^2 \geq \\ &\geq 2d \frac{\|x\|^2}{\varphi^2(t)} - 2c \frac{\|x\|}{\varphi(t)} \end{aligned}$$

and equality

$$F(r) - F(r_0) = 2 \int_{r_0}^r \frac{ds^2 - cs}{ms + 1} ds,$$

it follows that $\dot{W}_f(t, x) > 2(dr_0^2 - cr_0) > 0$ and $|\dot{V}_f(t, x)| \leq \dot{W}_f(t, x)$ once $\|x/\varphi(t)\| > r_0 > c/d$, or, equivalently, $V(t, x) > 0$.

Next, it is easily seen that in our case

$$w^0(t) = r_0^2 \lambda_C^+(t), \quad w_0(t) = r_0^2 \lambda_C^-(t).$$

If we pick w_*, w^* in such a way that

$$w_* < r_0^2 \inf_{t \in \mathbb{R}} \lambda_C^-(t), \quad w^* > r_0^2 \sup_{t \in \mathbb{R}} \lambda_C^+(t),$$

then, in view of Lemma 2, to satisfy the conditions (A),(B),(C) it is sufficient to define

$$\mathcal{W} := W^{-1}((w_*, w^*)), \quad \mathcal{M}_t := W_t^{-1}([0, w^*]) \cap \mathbb{L}_+(t).$$

Note, that in our case $c_* = c^* = 1$ and $\alpha(t) \geq 2(dr_0^2 - cr_0) > 0$.

Lastly, from (b) it follows that $\inf_{t \in \mathbb{R}} \lambda_{C, \min}^+(t) := \sigma_1 > 0$. Hence, $\frac{\|x\|^2}{\varphi^2(t)} \leq \frac{w^*}{\sigma_1}$ for all $t \in \mathbb{R}$, all $x \in \mathcal{M}_t$, and this yields $\nu < \infty$. Now, by the Theorem 1, there exists a solution $x_*(t)$, $t \in \mathbb{R}$, of the system (13) such that

$$V(t, x_*(t)) \leq \frac{r_0^2}{2} \left[\sup_{s \geq t} \lambda_C^+(s) - \inf_{s \leq t} \lambda_C^-(s) \right].$$

The estimate (14) is easily obtained by letting r_0 tend to c/d .

In order to prove the uniqueness of $x_*(t)$, it remains only to apply the Theorem 2 in the case where $U(t, x, y) := W(t, x - y)$, $V = V_+(t, x) := \|x\|^2/\varphi^2(t)$, $\eta(u) := u$, $\beta(t, r) := (1 - 2(k + l))/c$, $b(t, r) := 4cr$. \square

Remark 4. The number r_* does not exceed the largest root of the quadratic equation

$$\frac{d}{m} r^2 - 2 \left(\frac{c}{m} + \frac{d}{m^2} \right) r - F \left(\frac{c}{d} \right) - \frac{c^2}{2d^2} \sup_{t \in \mathbb{R}} \left[\sup_{s \geq t} \lambda_C^+(s) - \inf_{s \leq t} \lambda_C^-(s) \right] = 0.$$

Remark 5. The assertion of the Theorem 3 remains true if we require that the function $g(\cdot)$ is defined and satisfies the condition (c) not on the whole \mathbb{R}^{1+n} but only on a domain Ω which contains the set $W^{-1}([w_*, w^*]) \cap V^{-1}([0, V^*])$ where V-W-pair is defined by (15) and $V^* = w^* + \max\{\nu, -w_*\}$.

Example 1. Consider the following singular boundary value problem for scalar second order differential equation

$$\frac{d}{dt} \left(\frac{\dot{z}}{\rho(t)} \right) - \omega(t)z = Z(t, z, \dot{z}), \tag{16}$$

$$z(-\infty) = z(+\infty) = 0, \tag{17}$$

where $\rho(\cdot) \in C^1(\mathbb{R} \mapsto (0, \infty))$, $\omega(\cdot) \in C(\mathbb{R} \mapsto (0, \infty))$ are bounded functions and the function $Z(\cdot) \in C(\mathbb{R}^3 \mapsto \mathbb{R})$ satisfies a global Lipschitz condition: there exists a constant ℓ such that

$$|Z(t, x_1, y_1) - Z(t, x_2, y_2)| \leq \ell \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ \forall \{t, x_1, y_1, x_2, y_2\} \subset \mathbb{R}.$$

Let us show that if there exists a function $\varphi(\cdot) \in C^1(\mathbb{R} \mapsto (0, \infty))$ such that

$$|Z(t, 0, 0)| \leq \varphi(t), \quad \lim_{|t| \rightarrow \infty} \varphi(t) = 0, \quad \sup_{t \in \mathbb{R}} \frac{|\dot{\varphi}(t)|}{\varphi(t)} := l < \infty$$

and

$$k + l < \delta,$$

where

$$\delta := \min \left\{ \inf_{t \in \mathbb{R}} \rho(t), \inf_{t \in \mathbb{R}} \omega(t) \right\}, \quad k := \ell \max \left\{ 1, \sup_{t \in \mathbb{R}} \rho(t) \right\},$$

then the problem (16)–(17) has a unique solution $z_*(t) = O(\varphi(t))$.

By letting $x_1 = z$, $x_2 = \dot{z}/\rho(t)$, the equation (16) becomes equivalent to 2-D system of the form (13) with

$$A(t) = \begin{pmatrix} 0 & \rho(t) \\ \omega(t) & 0 \end{pmatrix}, \quad g(t, x) = \begin{pmatrix} 0 \\ Z(t, x_1, \rho(t)x_2) \end{pmatrix}.$$

Set $\langle C(t)x, x \rangle = x_1x_2/\delta$. Obviously this is a nondegenerate indefinite quadratic form of Morse index 1. One can easily show that

$$\|C(t)\| = 1/(2\delta) =: c, \quad \frac{d}{dt} \langle C(t)x, x \rangle_{A(t)x} \geq \|x\|^2,$$

$$c(k + l) < 1/2, \quad \|g(t, 0)\| \leq \varphi(t), \quad \|g(t, x) - g(t, y)\| \leq k\|x - y\|.$$

Now the unique solvability of the problem (16)–(17) in the class of functions $z(t) = O(\varphi(t))$ follows from the Theorem 3.

Note that if we slightly simplify our task by replacing the condition (17) with $\sup_{t \in \mathbb{R}} |z(t)| < \infty$, then the sufficient condition for solvability of the corresponding problem takes the form

$$\sup_{t \in \mathbb{R}} |Z(t, 0, 0)| < \infty, \quad k < \delta$$

(obviously, in this case $\varphi(t) \equiv \text{const}$, and $l = 0$). At the same time, by applying results of [18] combined with estimates for Green function derived in [34, 35], we can only obtain a rougher condition

$$2k\delta^{-3/2} \sqrt{\max \left\{ \sup_{t \in \mathbb{R}} \rho(t), \sup_{t \in \mathbb{R}} \omega(t) \right\}} < 1$$

(note that the expression under the square root is not less than δ).

Now let us lay down sufficient conditions for the existence of V_+ -bounded solutions in the case where $f(\cdot) \in C(\mathbb{R}^{1+n} \mapsto \mathbb{R}^n)$ is essentially nonlinear, e.g., $\|f(t, x)\|/\|x\| \rightarrow \infty, x \rightarrow \infty$. We are going to construct a V-W-pair in the form $V(t, x) = F(V_+(t, x)), W(t, x) = \langle S(t)x, x \rangle, V_+(t, x) = \langle B(t)x, x \rangle$ under the following conditions:

- (d): for any $t \in \mathbb{R}$, the operator $B(t)$ is positively definite and there exist projectors $P_+(t), P_-(t)$ on invariant subspaces $\mathbb{L}_+(t), \mathbb{L}_-(t)$ of operator $S(t)$ such that the restriction of $S(t)$ on $\mathbb{L}_+(t)$ (on $\mathbb{L}_-(t)$) is a positively definite (negatively definite) operator.
- (e): there exist functions $\gamma(\cdot) \in C(\mathbb{R} \mapsto (0, \infty)), \Gamma(\cdot) \in C((0, \infty) \mapsto \mathbb{R}), \Delta(\cdot) \in C((0, \infty) \mapsto (0, \infty))$ such that

$$\begin{aligned} \min_{\{x \in \mathbb{R}^n: \langle B(t)x, x \rangle = v\}} \langle S(t)f(t, x), x \rangle &\geq \gamma(t)\Gamma(v) \quad \forall v > 0, \\ \max_{\{x \in \mathbb{R}^n: \langle B(t)x, x \rangle = v\}} |\langle B(t)f(t, x), x \rangle| &\leq \gamma(t)\Delta(v) \quad \forall v > 0, \end{aligned}$$

and

$$\int_{-\infty}^0 \gamma(t) dt = \int_0^{\infty} \gamma(t) dt = \infty.$$

(f): the following inequalities hold true

$$\begin{aligned} \sup_{t \in \mathbb{R}} \lambda^+(t) < \infty, \quad \inf_{t \in \mathbb{R}} \lambda_-(t) > -\infty, \quad \limsup_{t \rightarrow -\infty} \lambda_-^+(t) > 0, \\ \inf_{t \in \mathbb{R}} \frac{\mu_-(t)}{\gamma(t)} =: \xi > -\infty, \quad \sup_{t \in \mathbb{R}} \frac{M(t)}{\gamma(t)} =: \varsigma < \infty, \end{aligned}$$

where $\lambda^+(t)$ and $\lambda_-(t)$ are, respectively, the maximal and the minimal characteristic values of the pencil $S(t) - \lambda B(t)$, $\lambda_-^+(t)$ is the minimal characteristic value of the pencil $P_+(t) [S(t) - \lambda B(t)]|_{\mathbb{L}_+(t)}$, $M(t)$ is the maximum of moduli of maximal and minimal characteristic values of the pencil $\dot{B}(t) - \mu B(t)$, and $\mu_-(t)$ is the minimal characteristic value of the pencil $\dot{S}(t) - \mu B(t)$.

(g): there exists a number $v_0 > 0$ such that

$$\begin{aligned} 2\Gamma(v_0) + \xi v_0 > 0, \quad \frac{\Gamma(v) - \Gamma(v_0)}{v - v_0} \geq -\frac{\xi}{2} \quad \forall v > v_0, \\ \int_{v_0}^{\infty} \frac{2\Gamma(v) + \xi v}{2\Delta(v) + \varsigma v} dv = \infty. \end{aligned}$$

We have the following result.

Theorem 4. *Let the system (1) satisfies in $\Omega := \mathbb{R}^{1+n}$ the conditions (d)–(g). Then there exists a solution $x_*(t)$ of this system such that*

$$\langle B(t)x_*(t), x_*(t) \rangle \leq F^{-1} \left(\frac{v_0}{2} \left[\sup_{s \geq t} \lambda^+(s) - \inf_{s \leq t} \lambda_-(s) \right] \right) \leq v_* \quad \forall t \in \mathbb{R},$$

where

$$F(v) := \int_{v_0}^v \frac{2\Gamma(u) + \xi u}{2\Delta(u) + \varsigma u} du$$

and v_* is the root of the equation

$$F(v) = \frac{v_0}{2} \left[\sup_{t \in \mathbb{R}} \lambda^+(t) - \inf_{t \in \mathbb{R}} \lambda_-(t) \right].$$

If in addition $2\gamma(t) + \mu_-(t) > 0$ for all $t \in \mathbb{R}$,

$$\langle S(t)(f(t, x + y) - f(t, x)), y \rangle \geq \gamma(t) \langle B(t)y, y \rangle \quad \forall (t, x, y) \in \mathbb{R}^{1+2n},$$

and

$$\int_0^{\pm\infty} \frac{2\gamma(s) + \mu_-(s)}{\max\{\lambda^+(s), |\lambda_-(s)|\}} ds = \pm\infty, \quad \liminf_{t \rightarrow \pm\infty} \frac{\ln \max\{\lambda^+(t), |\lambda_-(t)|\}}{\left| \int_0^t \frac{2\gamma(s) + \mu_-(s)}{\max\{\lambda^+(s), |\lambda_-(s)|\}} ds \right|} < 1,$$

then $x_*(t)$ is a unique solution of the system (1) satisfying the condition

$$\sup_{t \in \mathbb{R}} \langle B(t)x_*(t), x_*(t) \rangle < \infty.$$

Proof. Put $W(t, x) := \langle S(t)x, x \rangle$, $V_+(t, x) := \langle B(t)x, x \rangle$. Since

$$M(t) = \max_{\{x \in \mathbb{R}^n : V_+(t, x) = 1\}} \left| \langle \dot{B}(t)x, x \rangle \right|, \quad \mu_-(t) = \min_{\{x \in \mathbb{R}^n : V_+(t, x) = 1\}} \langle \dot{S}(t)x, x \rangle$$

(see, e.g., [36]), then

$$\begin{aligned} \left| \left[\dot{V}_+(t, x) \right]_f \right| &\leq 2\gamma(t)\Delta(V_+(t, x)) + M(t)V_+(t, x) \leq \\ &\leq \gamma(t) (2\Delta(V_+(t, x)) + \varsigma V_+(t, x)), \\ \dot{W}_f(t, x) &\geq 2\gamma(t)\Gamma(V_+(t, x)) + \mu_-(t)V_+(t, x) \geq \\ &\geq \gamma(t) (2\Gamma(V_+(t, x)) + \xi V_+(t, x)), \end{aligned}$$

once $V_+(t, x) > v_0$, and it is naturally to define in this case

$$V(t, x) = F(V_+(t, x)). \tag{18}$$

Obviously that the inequality (2) and condition (B) are satisfied with $c_* = c^* = 1$, $\alpha(t) \geq \gamma(t)(2\Gamma(v_0) + \xi v_0)$.

Taking into account that the function $W_t(x)$ has the unique critical point $x = 0$, we have

$$\begin{aligned} w^0(t) &:= \max_{\{x \in \mathbb{R}^n : V_+(t, x) \leq v_0\}} W(t, x) = \lambda^+(t)v_0, \\ w_0(t) &:= \min_{\{x \in \mathbb{R}^n : V_+(t, x) \leq v_0\}} W(t, x) = \lambda_-(t)v_0. \end{aligned}$$

If we choose numbers w_*, w^* in such a way that

$$w_* < \omega_0 = \inf_{t \in \mathbb{R}} \lambda_-(t)v_0, \quad w^* > \omega^0 = \sup_{t \in \mathbb{R}} \lambda^+(t)v_0, \tag{19}$$

and define $\mathcal{W} := W^{-1}((w_*, w^*))$, then the condition (A) will be satisfied.

As has been already shown in proof of Theorem 3 the family of sets

$$\mathcal{M}_t := W_t^{-1}([0, w^*]) \cap \mathbb{L}_+(t)$$

satisfy the condition (C). Now to prove the existence of V -bounded solution it remains only to show that $\nu < \infty$. It is easily seen that

$$\begin{aligned} \min\{W_t(x) : x \in \mathcal{M}_t, V_+(t, x) > v_0\} &= \lambda_-^+(t)v_0 > 0, \\ \max_{\{x \in \mathcal{M}_t\}} V_+(t, x) &= \frac{w^*}{\lambda_-^+(t)}, \end{aligned}$$

and in view of condition (f) we have $\liminf_{t \rightarrow -\infty} (w^*/\lambda_-^+(t)) < \infty$. Hence, $\nu < \infty$.

In order to prove the uniqueness of V_+ -bounded solution of the system (1), introduce the function $U(t, x, y) := \langle S(t)(x - y), (x - y) \rangle$. It is easily seen that

$$\begin{aligned} \dot{U}_{(f,f)}(t, x, y) &\geq (2\gamma(t) + \mu_-(t)) \langle B(t)(x - y), x - y \rangle \geq \\ &\geq \frac{2\gamma(t) + \mu_-(t)}{\max\{\lambda^+(t), |\lambda_-(t)|\}} |U(t, x, y)|, \\ |U(t, x, y)| &\leq \max\{\lambda^+(t), |\lambda_-(t)|\} \langle B(t)(x - y), x - y \rangle. \end{aligned}$$

Now the uniqueness result follows from Theorem 2 if we define

$$\begin{aligned} \beta(t, r) &:= \frac{2\gamma(t) + \mu_-(t)}{\max\{\lambda^+(t), |\lambda_-(t)|\}}, \\ b(t, r) &:= 4 \max\{\lambda^+(t), |\lambda_-(t)|\} r, \quad \eta(u) := u. \end{aligned}$$

□

5 V -bounded solutions of Lagrangian systems

Consider a natural Lagrangian system subjected to smooth time-varying holonomic constraint. The Lagrangian of such a system can be represented in the form

$$L(t, q, \dot{q}) := \frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \langle a(t, q), \dot{q} \rangle + \Phi(t, q) \tag{20}$$

where $q = (q_1, \dots, q_m) \in \mathbb{R}^m$ are generalized coordinates, $\mathcal{A}(\cdot) : \mathbb{R}^{1+m} \mapsto \text{Aut}(\mathbb{R}^m)$, $a(\cdot) : \mathbb{R}^{1+m} \mapsto \mathbb{R}^m$, $\Phi(\cdot) : \mathbb{R}^{1+m} \mapsto \mathbb{R}$ are C^2 -mappings, and besides, $\mathcal{A}(\cdot)$ takes values in the space of positive-definite operators. Our goal is to show that if the Lagrangian has certain directional quasiconvexity property, namely

(α): there exist positive numbers κ, R and a function $\Psi(\cdot) : \mathbb{R}^{1+m} \mapsto \mathbb{R}_+$ such that from

$$\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \geq R$$

it follows that

$$\frac{\partial L}{\partial q_i} q_i + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \geq \kappa \left(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \right) \quad (21)$$

(summation over repeating indices),

then under some additional technical growth conditions imposed on $\mathcal{A}(\cdot), a(\cdot), \Psi(\cdot)$ the Lagrangian system possesses a global solution along which the function $\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q)$ is bounded.

Remark 6. It is easily seen that the inequality (21) yields

$$\left\langle \left(\mathcal{A}(t, q) + \frac{1}{2} \frac{\partial \mathcal{A}(t, q)}{\partial q_i} q_i \right) y, y \right\rangle \geq \frac{\kappa}{4} \langle \mathcal{A}(t, q) y, y \rangle \quad \forall (t, q, y) \in \mathbb{R}^{1+2n}. \quad (22)$$

It should be also noted that the Assumptions (H4), (H5) in [26] implies that

$$\frac{\partial L}{\partial q_i} q_i + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \geq \kappa (\|\dot{q}\|^2 + \|q\|^2)$$

once $\|\dot{q}\|^2 + \|q\|^2$ is sufficiently large.

In what follows, we shall also assume that:

(β): there exists a nondecreasing coercive functions $\underline{\Theta}(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+, \bar{\Theta}(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such

$$\underline{\Theta}(\Psi(t, q)) \leq \langle \mathcal{A}(t, q) q, q \rangle \leq \bar{\Theta}(\Psi(t, q)) \quad \forall (t, q) \in \mathbb{R}^{1+m}.$$

(γ): there exist numbers $\theta \in [0, 1]$ and $K > 0$ such that from

$$\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \geq R$$

it follows that

$$\begin{aligned} \left| \frac{1}{2} \left\langle \frac{\partial \mathcal{A}(t, q)}{\partial t} \dot{q}, \dot{q} \right\rangle + \frac{\partial(\Phi(t, q) + \Psi(t, q))}{\partial q_i} \dot{q}_i + \frac{\partial \Psi(t, q)}{\partial t} \right| &\leq \\ &\leq K \left(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \right)^{\theta+1}, \end{aligned}$$

R being the same number as in (α).

(δ): there exist a nondecreasing function $\Xi(\cdot) : \mathbb{R} \mapsto \mathbb{R}_+$ such that

$$\max_{\|y\|=1} \frac{|\langle a(t, q), y \rangle|}{\sqrt{\langle \mathcal{A}(t, q)y, y \rangle}} \leq \Xi(\Psi(t, q)) \quad \forall (t, q) \in \mathbb{R}^{1+m}.$$

In order to apply the results of Section 3, introduce the functions

$$\begin{aligned} W(t, q, \dot{q}) &:= \langle \mathcal{A}(t, q)\dot{q} + a(t, q), q \rangle, \\ V(t, q, \dot{q}) &:= \bar{V} \left(\frac{1}{2} \langle \mathcal{A}(t, q)\dot{q}, \dot{q} \rangle + \Psi(t, q) \right), \end{aligned} \tag{23}$$

where $\bar{V}(\cdot) \in C^1(\mathbb{R} \mapsto (-1, \infty))$ is a strictly increasing function which for $r \geq R$ is defined as

$$\bar{V}(r) =: \begin{cases} \ln(r/R) & \text{if } \theta = 1, \\ (r^{1-\theta} - R^{1-\theta}) / (1 - \theta) & \text{if } \theta \in [0, 1). \end{cases}$$

Lemma 3. *From $V(t, q, \dot{q}) \geq 0$ it follows that*

$$\left| \dot{V}_f(t, q, \dot{q}) \right| \leq \frac{K}{\kappa} \dot{W}_f(t, q, \dot{q}) \quad \text{and} \quad \dot{W}_f(t, q, \dot{q}) \geq \kappa R,$$

where $f(t, q, \dot{q}) := \left(\dot{q}, \left(\frac{\partial^2 L}{\partial \dot{q}^2} \right)^{-1} \left(\frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial t \partial \dot{q}} - \frac{\partial^2 L}{\partial \dot{q} \partial q_i} \dot{q}_i \right) \right)$ is the vector field generated in the phase space \mathbb{R}^{2m} by the Lagrangian system.

Proof. Note that $W = \frac{\partial L}{\partial \dot{q}_i} q_i$. The equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

yields

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} q_i \Big|_f = \frac{\partial L}{\partial q_i} q_i + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i.$$

Obviously,

$$V(t, q, \dot{q}) \geq 0 \quad \Leftrightarrow \quad \frac{1}{2} \langle \mathcal{A}(t, q)\dot{q}, \dot{q} \rangle + \Psi(t, q) \geq R.$$

Then by assumption (α) we have

$$\dot{W}_f(t, q, \dot{q}) \geq \kappa \left(\frac{1}{2} \langle \mathcal{A}(t, q)\dot{q}, \dot{q} \rangle + \Psi(t, q) \right) \geq \kappa R \quad \text{once} \quad V(t, q, \dot{q}) \geq 0. \tag{24}$$

In order to estimate the function $\dot{V}_f(\cdot)$, introduce the Hamiltonian in a standard way:

$$H(t, q, \dot{q}) = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = \frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle - \Phi(t, q).$$

As is well known, $\frac{dH}{dt} = \frac{\partial H}{\partial t}$, hence

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \right) &= \frac{dH(t, q, \dot{q})}{dt} + \frac{d\Phi(t, q)}{dt} + \frac{d\Psi(t, q)}{dt} = \\ &= \frac{1}{2} \left\langle \frac{\partial \mathcal{A}(t, q)}{\partial t} \dot{q}, \dot{q} \right\rangle + \frac{\partial (\Phi(t, q) + \Psi(t, q))}{\partial q_i} \dot{q}_i + \frac{\partial \Psi(t, q)}{\partial t}. \end{aligned}$$

By assumption (γ) , if $V(t, q, \dot{q}) \geq 0$, then

$$\begin{aligned} & \left| \dot{V}_f(t, q, \dot{q}) \right| = \\ &= \left(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \right)^{-\theta} \left| \frac{d}{dt} \left(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \right) \right| \leq \\ &\leq K \left(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \right) \leq \frac{K}{\kappa} \dot{W}_f(t, q, \dot{q}). \end{aligned}$$

□

Lemma 4. For the functions $V(\cdot)$ and $W(\cdot)$ defined by (23), the corresponding functions $w_0(\cdot)$, $w^0(\cdot)$ defined by (3) satisfy the following estimates:

$$w_0(t) \geq \tilde{w}_0(t) := \min_{q \in \Psi_t^{-1}([0, R])} \left\{ \langle a(t, q), q \rangle - \sqrt{2[R - \Psi(t, q)] \langle \mathcal{A}(t, q) q, q \rangle} \right\}, \quad (25)$$

$$w^0(t) \leq \tilde{w}^0(t) := \max_{q \in \Psi_t^{-1}([0, R])} \left\{ \langle a(t, q), q \rangle + \sqrt{2[R - \Psi(t, q)] \langle \mathcal{A}(t, q) q, q \rangle} \right\}, \quad (26)$$

$$\omega_0 := \inf_{t \in \mathbb{R}} w_0(t) \geq - \max_{s \in [0, R]} \sqrt{\Theta(s)} \left[\sqrt{2(R - s)} + \Xi(s) \right], \quad (27)$$

$$\omega^0 := \sup_{t \in \mathbb{R}} w^0(t) \leq \max_{s \in [0, R]} \sqrt{\Theta(s)} \left[\sqrt{2(R - s)} + \Xi(s) \right]. \quad (28)$$

Proof. We know that

$$V_t^{-1}(0) = \{(q, \dot{q}) \in \mathbb{R}^{2m} : \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle = 2[R - \Psi(t, q)], \Psi_t(q) \leq R\}$$

and

$$\langle a(t, q), q \rangle - |\langle \mathcal{A}(t, q)\dot{q}, q \rangle| \leq W(t, q, \dot{q}) \leq \langle a(t, q), q \rangle + |\langle \mathcal{A}(t, q)\dot{q}, q \rangle|.$$

By assumptions (β) and (δ) we have

$$|\langle a(t, q), q \rangle| \leq \Xi(\Psi(t, q))\sqrt{\Theta(\Psi(t, q))}.$$

Now to obtain the required estimates it is sufficiently to observe that

$$\begin{aligned} |\langle \mathcal{A}(t, q)\dot{q}, q \rangle| \Big|_{V_t^{-1}(0)} &\leq \sqrt{\langle \mathcal{A}(t, q)\dot{q}, \dot{q} \rangle \langle \mathcal{A}(t, q)q, q \rangle} \Big|_{V_t^{-1}(0)} \leq \\ &\leq \sqrt{2[R - \Psi(t, q)]\bar{\Theta}(\Psi(t, q))} \end{aligned}$$

and $\Psi(t, q) \geq 0$. □

Now we are in position to prove the following theorem.

Theorem 5. *Let for the Lagrangian (20) the assumptions (α) – (δ) be valid. Then the corresponding Lagrangian system has a global solution $q_*(t)$ which for some positive number $\sigma \in (0, (\omega^0 - \omega_0)/(\kappa R)]$ satisfies the inequalities*

$$\begin{aligned} &\frac{1}{2} \langle \mathcal{A}(t, q_*(t)) \dot{q}_*(t), \dot{q}_*(t) \rangle + \Psi(t, q_*(t)) \leq \\ &\leq \mathfrak{f}_{\theta, R} \left(\frac{K}{2\kappa} \left[\sup_{t \leq s \leq t+\sigma} \tilde{w}^0(s) - \inf_{t-\sigma \leq s \leq t} \tilde{w}_0(s) \right] \right), \\ &\omega_0 \leq \langle \mathcal{A}(t, q)\dot{q} + a(t, q), q \rangle \leq \omega^0, \end{aligned}$$

where

$$\mathfrak{f}_{\theta, R}(z) := \begin{cases} R e^z & \text{if } \theta = 1, \\ [(1 - \theta)z + R^{1-\theta}]^{\frac{1}{1-\theta}} & \text{if } \theta \in [0, 1), \end{cases}$$

and the functions $\tilde{w}_0(t)$, $\tilde{w}^0(t)$ and numbers ω_0 , ω^0 are defined by (25)–(28).

Proof. Let $w_* < \omega_0$ and $w^* > \omega^0$ be arbitrary numbers, where ω_0 , ω^0 are defined by (27), (28). The function $W(\cdot)$ in new coordinates q , $p := \mathcal{A}(t, q)\dot{q} + a(t, q)$ takes the form of an indefinite nondegenerate quadratic form $\langle p, q \rangle$. From this it follows that the set $\mathcal{W} := W^{-1}((w_*, w^*))$ is connected and for each $t \in \mathbb{R}$ the function $W_t(\cdot)$ restricted to the set $V_t^{-1}((-\infty, 0])$ takes its maximal and minimal values on the boundary $V_t^{-1}(0)$. Hence, $V^{-1}((-\infty, 0]) \subset \mathcal{W}$ and by Lemma 3 the conditions (A) and (B) are valid

with $c_* = c^* = K/\kappa$ and $\alpha(t) \geq \kappa R$ respectively. Obviously, the functions $\tau_{\pm}(t)$ defined in Theorem 1 satisfy in our case the inequalities

$$|\tau_{\pm}(t) - t| \leq \frac{\omega^0 - \omega_0}{\kappa R}.$$

Now we define the set

$$\mathcal{M}_t := \{(q, \dot{q}) \in \mathbb{R}^{2m} : \dot{q} = q - \mathcal{A}^{-1}(t, q)a(t, q), \langle \mathcal{A}(t, q)q, q \rangle \leq w^*\}.$$

Obviously, $0 \leq W_t(q, \dot{q})|_{\mathcal{M}_t} = \langle \mathcal{A}(t, q)q, q \rangle \leq w^*$. Since by assumption (β) $\langle \mathcal{A}(t, q)q, q \rangle$ is spatially coercive, and (22) implies that

$$\frac{\partial \langle \mathcal{A}(t, q)q, q \rangle}{\partial q_i} q_i = \left\langle \left(2\mathcal{A}(t, q) + \frac{\partial \mathcal{A}(t, q)}{\partial q_i} q_i \right) q, q \right\rangle \geq \frac{\kappa}{2} \langle \mathcal{A}(t, q)q, q \rangle, \quad (29)$$

then $\langle \mathcal{A}(t, q)q, q \rangle$ is regular spatially coercive. For this reason, \mathcal{M}_t is a compact manifold with boundary.

In order to show that ν defined by (8) is bounded, note that in view of assumption (β) the function $\Psi(t, q)$ is bounded from above by the constant $\underline{\Theta}^{-1}(w^*)$ on the set where $\langle \mathcal{A}(t, q)q, q \rangle \leq w^*$, and now, taking into account the definition of $V(\cdot)$, it is sufficient to prove that

$$\sup_{t \in \mathbb{R}} \max \{ \langle \mathcal{A}(t, q)\dot{q}, \dot{q} \rangle : \dot{q} = q - \mathcal{A}^{-1}(t, q)a(t, q), \langle \mathcal{A}(t, q)q, q \rangle \leq w^* \} < \infty. \quad (30)$$

But from (δ) for such points that $\langle \mathcal{A}(t, q)q, q \rangle \leq w^*$, we obtain

$$\begin{aligned} |\langle a(t, q), q \rangle| &\leq \sqrt{w^*} \Xi(\underline{\Theta}^{-1}(w^*)), \\ \sqrt{\langle \mathcal{A}^{-1}(t, q)a(t, q), a(t, q) \rangle} &\leq \Xi(\underline{\Theta}^{-1}(w^*)). \end{aligned}$$

Hence,

$$\begin{aligned} &\langle \mathcal{A}(t, q)[q - \mathcal{A}^{-1}(t, q)a(t, q)], [q - \mathcal{A}^{-1}(t, q)a(t, q)] \rangle = \\ &= \langle \mathcal{A}(t, q)q, q \rangle - 2\langle a(t, q), q \rangle + \langle \mathcal{A}^{-1}a(t, q), a(t, q) \rangle \leq \\ &\leq w^* + 2\sqrt{w^*} \Xi(\underline{\Theta}^{-1}(w^*)) + \Xi^2(\underline{\Theta}^{-1}(w^*)), \end{aligned}$$

and (30) is proved.

Let us show that the condition (C) is valid. Since in (q, p) -coordinates the function $W(q, p) = \langle p, q \rangle$ does not depend on t , it remains only to prove that for any fixed $t \in \mathbb{R}$ the set

$$\partial \mathcal{M}_t = \{(q, p) \in \mathbb{R}^{2m} : p = \mathcal{A}(t, q)q, \langle \mathcal{A}(t, q)q, q \rangle = w^*\}$$

is a retract of $W_t^{-1}(w^*) = \{(q, p) \in \mathbb{R}^{2m} : \langle p, q \rangle = w^*\}$. Observe that for any $q \neq 0$ from (29) we get

$$\frac{d}{d\tau} e^{2\tau} \langle \mathcal{A}(t, e^\tau q) q, q \rangle \geq \frac{\kappa}{2} e^{2\tau} \langle \mathcal{A}(t, e^\tau q) q, q \rangle.$$

This implies that for any fixed q the mapping $\tau \mapsto e^{2\tau} \langle \mathcal{A}(t, e^\tau q) q, q \rangle$ is a diffeomorphism of \mathbb{R} onto $(0, \infty)$. Hence, for any $(q, z) \in \mathbb{R}^m \times (0, \infty)$ there exists a unique $\tau(q, z)$ such that

$$e^{2\tau} \langle \mathcal{A}(t, e^\tau q) q, q \rangle \Big|_{\tau=\tau(q,z)} = z, \quad \tau(q, \langle \mathcal{A}(t, q) q, q \rangle) = 0.$$

By the inverse function theorem the mapping $\tau(\cdot) : (\mathbb{R}^m \setminus \{0\}) \times (0, \infty) \mapsto \mathbb{R}$ which we have constructed is smooth. Now the required retraction is defined by the mapping

$$q \mapsto e^{\tau(q, \langle p, q \rangle)} q, \quad p \mapsto e^{\tau(q, \langle p, q \rangle)} \mathcal{A}\left(t, e^{\tau(q, \langle p, q \rangle)} q\right) q.$$

Now the existence of searched solution $q_*(t)$ follows from Theorem 1. □

Corollary 1. *If the assumptions (α) – (δ) are valid with $\Psi(\cdot) = \Phi(\cdot)$, then the solution $q_*(t)$ has the property $\sup_{t \in \mathbb{R}} |L(t, q_*(t), \dot{q}_*(t))| < \infty$.*

The next two lemmas will be useful for verifying the assumptions (α) and (γ) .

Lemma 5. *Let there exist positive constants κ, R_0, c_1, c_2 such that*

$$\min_{\|y\|=1} \frac{\left\langle \left(\mathcal{A}(t, q) + \frac{1}{2} \frac{\partial \mathcal{A}(t, q)}{\partial q_i} q_i \right) y, y \right\rangle}{\langle \mathcal{A}(t, q) y, y \rangle} \geq \kappa > 0 \quad \forall (t, q) \in \mathbb{R}^{1+m}, \quad (31)$$

$$\frac{\partial \Phi(t, q)}{\partial q_i} q_i \geq \kappa \Psi(t, q) + \frac{1}{2\kappa} \max_{\|y\|=1} \frac{\left\langle a(t, q) + \frac{\partial a(t, q)}{\partial q_i} q_i, y \right\rangle^2}{\langle \mathcal{A}(t, q) y, y \rangle} \quad (32)$$

once $\Psi(t, q) \geq R_0$,

$$\frac{\partial \Phi(t, q)}{\partial q_i} q_i \geq -c_1, \quad \max_{\|y\|=1} \frac{\left| \left\langle \frac{\partial a(t, q)}{\partial q_i} q_i, y \right\rangle \right|}{\sqrt{\langle \mathcal{A}(t, q) y, y \rangle}} \leq c_2 \quad (33)$$

once $\Psi(t, q) \leq R_0$;

Then under the assumption (δ) , the assumption (α) is valid with

$$R := R_0 + \left[\frac{\sqrt{2}(c_2 + \Xi(R_0)) + \sqrt{2(c_2 + \Xi(R_0))^2 + 4\kappa(c_1 + \kappa R_0)}}{2\kappa} \right]^2.$$

Proof. From (31) it follows that

$$\frac{\partial L}{\partial q_i} q_i + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \geq \kappa \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \left\langle a(t, q) + \frac{\partial a(t, q)}{\partial q_i} q_i, \dot{q} \right\rangle + \frac{\partial \Phi(t, q)}{\partial q_i} q_i.$$

If we put $y = \|\dot{q}\|^{-1} \dot{q} \in \mathbb{S}_1(0) := \{y \in \mathbb{R}^m : \|y\| = 1\}$ and $z := \sqrt{\langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle} / \sqrt{2}$, then it is sufficient to show that the inequality

$$z^2 + \Psi(t, q) \geq R$$

yields

$$\kappa z^2 - \sqrt{2} \frac{\left| \left\langle a(t, q) + \frac{\partial a(t, q)}{\partial q_i} q_i, y \right\rangle \right|}{\sqrt{\langle \mathcal{A}(t, q) y, y \rangle}} z + \frac{\partial \Phi(t, q)}{\partial q_i} q_i - \kappa \Psi(t, q) \geq 0.$$

But if $\Psi(t, q) \geq R_0$ then in view of (32) the quadratic polynomial (with respect to z) in the left-hand side of the last inequality takes only nonnegative values for all $y \in \mathbb{S}_1(0)$. And if $\Psi(t, q) \leq R_0$, then taking into account assumptions (33), (δ) , it is no hard to show that the greatest root of this polynomial (if it exists) does not exceed $\sqrt{R - R_0} \leq \sqrt{R - \Psi(t, q)}$ for all $y \in \mathbb{S}_1(0)$. Hence, in this case, the polynomial also takes nonnegative values for $z \geq \sqrt{R - \Psi(t, q)}$. \square

Lemma 6. *Let there exist a number $\theta \in [0, 1]$ and nonnegative numbers c_3, \dots, c_8 such that*

$$\max_{\|y\|=1} \frac{\left| \left\langle \frac{\partial \mathcal{A}(t, q)}{\partial t} y, y \right\rangle \right|}{\langle \mathcal{A}(t, q) y, y \rangle} \leq c_3 \Psi^\theta(t, q) + c_4 \quad \forall (t, q) \in \mathbb{R}^{1+m},$$

$$\max_{\|y\|=1} \frac{\left| \frac{\partial(\Phi(t, q) + \Psi(t, q))}{\partial q_i} y_i \right|}{\sqrt{\langle \mathcal{A}(t, q) y, y \rangle}} \leq c_5 \Psi^{\theta+1/2}(t, q) + c_6 \quad \forall (t, q) \in \mathbb{R}^{1+m},$$

$$\left| \frac{\partial \Psi(t, q)}{\partial t} \right| \leq c_7 \Psi^{\theta+1}(t, q) + c_8 \quad \forall (t, q) \in \mathbb{R}^{1+m}.$$

Then the assumption (γ) is valid with

$$K = c_3 + \sqrt{2}c_5 + c_7 + R^{-\theta} \left(c_4 + \sqrt{2}R^{-1/2}c_6 + R^{-1}c_8 \right).$$

Proof. Let again $z := \sqrt{\langle \mathcal{A}(t, q)\dot{q}, \dot{q} \rangle} / \sqrt{2}$. Then we have

$$\begin{aligned} & \left| \frac{1}{2} \left\langle \frac{\partial \mathcal{A}(t, q)}{\partial t} \dot{q}, \dot{q} \right\rangle + \frac{\partial(\Phi(t, q) + \Psi(t, q))}{\partial q_i} \dot{q}_i + \frac{\partial \Psi(t, q)}{\partial t} \right| \leq \\ & \leq \left(c_3 \Psi^\theta(t, q) + c_4 \right) z^2 + \sqrt{2} \left(c_5 \Psi^{\theta+1/2}(t, q) + c_6 \right) z + c_7 \Psi^{\theta+1}(t, q) + c_8 \leq \\ & \leq \left(c_3(z^2 + \Psi(t, q))^\theta + c_4 \right) (z^2 + \Psi(t, q)) + \\ & + \sqrt{2} \left(c_5(z^2 + \Psi(t, q))^{\theta+1/2} + c_6 \right) \sqrt{z^2 + \Psi(t, q)} + \\ & + c_7(z^2 + \Psi(t, q))^{\theta+1} + c_8 \leq K(z^2 + \Psi(t, q))^{\theta+1}. \end{aligned}$$

□

Let us now discuss the uniqueness problem. Usually, to guarantee the uniqueness of bounded solutions (in particular, almost periodic solutions) to Lagrangian systems, the convexity of Lagrangian function is required. In Cieutat's paper [26] it is assumed that the function $\frac{\partial L(t, \cdot)}{\partial u} : \mathbb{R}^{2m} \mapsto \mathbb{R}^{2m}$ is globally Lipschitzian with time independent Lipschitz constant, and the convexity condition is formulated as follows: there exists a constant $c > 0$ such that

$$\begin{aligned} & \left(\frac{\partial L(t, u)}{\partial u_i} - \frac{\partial L(t, v)}{\partial v_i} \right) (u_i - v_i) \geq c \|u - v\|^2 \tag{34} \\ & \forall u := (q', \dot{q}'), v := (q'', \dot{q}'') \in \mathbb{R}^{2m}. \end{aligned}$$

It should be noted that for Lagrangian (20), in the case where $\mathcal{A}(t, q)$ nonlinearly depends on q , the above global conditions look unnatural (see the Remark 7 below).

For Lagrangian (20), we are going to relax the conditions of [26] via the Theorem 2. (However, here for simplicity we consider the case where $\mathcal{A}(\cdot)$, $a(\cdot)$ and $\Phi(\cdot)$ are C^2 -mappings).

Put

$$\tilde{V}(t, u, v) := \max \left\{ \frac{1}{2} \langle \mathcal{A}(t, q') \dot{q}', \dot{q}' \rangle + \Psi(t, q'), \frac{1}{2} \langle \mathcal{A}(t, q'') \dot{q}'', \dot{q}'' \rangle + \Psi(t, q'') \right\},$$

and denote

$$N(t; r) := \sup \left\{ \frac{\left\| \frac{\partial L(t, u)}{\partial \dot{q}'} - \frac{\partial L(t, v)}{\partial \dot{q}''} \right\|}{\|u - v\|} : (u, v) \in \tilde{V}_t^{-1}((-\infty, r]), u \neq v \right\}.$$

Let $\lambda(t, q)$ and $\Lambda(t, q)$ be, respectively, the minimal and the maximal eigenvalues of operator $\mathcal{A}(t, q)$. Define

$$\vartheta(t; r) := \max \left\{ \frac{\Lambda(t, q'')}{\lambda(t, q')} : q' \in \Psi_t^{-1}([0, r]), q'' \in \Psi_t^{-1}([0, r]) \right\}.$$

For any set $\tilde{\Omega} \subset \mathbb{R}^{1+2m}$ we define the set

$$\tilde{\Omega}^* := \left\{ (t, u, v) \in \mathbb{R}^{1+4m} : (t, u) \in \tilde{\Omega}, (t, v) \in \tilde{\Omega} \right\}$$

(see Theorem 2).

Theorem 6. *Let the assumptions (β) and (δ) be valid and let for a set $\tilde{\Omega} \subset \mathbb{R}^{1+2m}$ there exist numbers $r > 0$ and $d \geq 1$ such that $\tilde{\Omega}^* \subseteq \tilde{V}^{-1}([0, r])$ and*

$$\varrho(t; r, d) := \inf \left\{ \frac{\left(\frac{\partial L(t, u)}{\partial u_i} - \frac{\partial L(t, v)}{\partial v_i} \right) (u_i - v_i)}{\|u - v\|^{2d}} : (u, v) \in \tilde{\Omega}_t^*, u \neq v \right\} > 0.$$

Suppose in addition that

$$I(t; r, d) := \left| \int_0^t \frac{\varrho(s; r, d)}{N^d(s; r)} ds \right| \rightarrow \infty, \quad t \rightarrow \pm\infty,$$

and if $d = 1$, then also

$$\liminf_{t \rightarrow \pm\infty} \frac{\ln \left(1 + \sqrt{\vartheta(t; r)} \right)}{I(t; r, 1)} < 1.$$

Then the Lagrangian system cannot have two different global solutions $q_j(t), t \in \mathbb{R}, j=1,2$, such that $(t, q_j(t), \dot{q}_j(t)) \in \tilde{\Omega}$ for all $t \in \mathbb{R}, j = 1, 2$.

Proof. In order to apply the Theorem 2, we introduce the function

$$\begin{aligned} U(t, u, v) &:= \left(\frac{\partial L(t, u)}{\partial \dot{q}'_i} - \frac{\partial L(t, v)}{\partial \dot{q}''_i} \right) (q'_i - q''_i) = \\ &= \langle \mathcal{A}(t, q') \dot{q}' + a(t, q') - \mathcal{A}(t, q'') \dot{q}'' - a(t, q''), q' - q'' \rangle. \end{aligned}$$

In the same way as in Lemma 4, one can show that

$$\begin{aligned} |\langle \mathcal{A}(t, q) \dot{q} + a(t, q), q \rangle| &\leq \sqrt{\Theta(\Psi(t, q))} \left[\sqrt{2[r - \Psi(t, q)]} + \Xi(\Psi(t, q)) \right] \\ &\quad \forall (t, q, \dot{q}) \in \tilde{\Omega}, \end{aligned}$$

and since $\langle \mathcal{A}(t, q')q'', q'' \rangle \leq \frac{\Lambda(t, q')}{\lambda(t, q'')} \bar{\Theta}(\Psi(t, q''))$, then

$$\begin{aligned} & |\langle \mathcal{A}(t, q')\dot{q}' + a(t, q'), q'' \rangle| \leq \\ & \leq \sqrt{\langle \mathcal{A}(t, q')q'', q'' \rangle} \left[\sqrt{\langle \mathcal{A}(t, q')\dot{q}', \dot{q}' \rangle} + \Xi(\Psi(t, q')) \right] \leq \\ & \leq \sqrt{\vartheta(t; r) \bar{\Theta}(\Psi(t, q''))} \left[\sqrt{2[r - \Psi(t, q')]} + \Xi(\Psi(t, q')) \right] \quad \forall (t, u, v) \in \tilde{\Omega}^*. \end{aligned}$$

Now we have

$$|U(t, u, v)| \leq 2\omega^*(r) \left[1 + \sqrt{\vartheta(t; r)} \right] \quad \forall (t, u, v) \in \tilde{\Omega}^*, \quad (35)$$

where

$$\omega^*(r) := \sqrt{\bar{\Theta}(r)} \max_{1 \leq s \leq r} \left[\sqrt{2[r - s]} + \Xi(s) \right].$$

Hence, in the case under consideration, the function $b(t, r)$ from Theorem 2 satisfies the inequality

$$b(t, r) \leq 2\omega^*(r) \left(1 + \sqrt{\vartheta(t; r)} \right).$$

Nextly, the inequality

$$|U(t, u, v)| \leq N(t; r) \|u - v\|^2$$

together with conditions imposed on $L(\cdot)$ yields

$$\begin{aligned} \dot{U}(t, u, v)_{(f, f)} &= \left(\frac{\partial L(t, u)}{\partial u_i} - \frac{\partial L(t, v)}{\partial v_i} \right) (u_i - v_i) \geq \varrho(t; r, d) \|u - v\|^{2d} \geq \\ & \beta(t, r) |U(t, u, v)|^d \end{aligned}$$

if $(t, u, v) \in \tilde{\Omega}^*$ where $\beta(t; r, d) := \varrho(t; r, d)N^{-d}(t, r)$. Now if we put $h(u) = \int_1^u s^{-d} ds$, then the reasoning which we used when proving the Theorem 2 yields the assertion of the Theorem 6. \square

It appears that instead of the convexity condition of Theorem 6 it is preferable to verify an analogous assumption for corresponding Hamiltonian

$$\begin{aligned} H(t, z) \equiv H(t, q, p) &:= \frac{1}{2} \langle \mathcal{A}^{-1}(t, q)(p - a(t, q)), p - a(t, q) \rangle - \Phi(t, q) \quad (36) \\ & (z := (q, p)). \end{aligned}$$

Let Id_m and 0_m be the identity matrix and the zero matrix of dimensions m respectively. Introduce the matrices

$$I := \begin{pmatrix} -\text{Id}_m & 0_m \\ 0_m & \text{Id}_m \end{pmatrix}, \quad J := \begin{pmatrix} 0_m & \text{Id}_m \\ -\text{Id}_m & 0_m \end{pmatrix}.$$

Put $z' := (q', p')$, $z'' := (q'', p'')$ and denote by $\hat{V}(t, z', z'')$ the function obtained from $\hat{V}(t, u, v)$ after the substitutions $\dot{q}' = \mathcal{A}^{-1}(t, q')(p' - a(t, q'))$, $\dot{q}'' = \mathcal{A}^{-1}(t, q'')(p'' - a(t, q''))$. Obviously,

$$\hat{V}(t, z', z'') := \max \left\{ H(t, z') + \Phi(t, q') + \Psi(t, q'), \right. \\ \left. H(t, z'') + \Phi(t, q'') + \Psi(t, q'') \right\}.$$

Theorem 7. *Let the assumptions (β) and (δ) be valid and let for a set $\hat{\Omega} \subset \mathbb{R}^{1+2m}$ there exist numbers $r > 0$ and $d \geq 1$ such that $\hat{\Omega}^* \subseteq \hat{V}^{-1}([0, r])$ and*

$$\hat{\varrho}(t; r, d) := \inf \left\{ \frac{\left\langle I \left(\frac{\partial H(t, z')}{\partial z'} - \frac{\partial H(t, z'')}{\partial z''} \right), z' - z'' \right\rangle}{\|z' - z''\|^{2d}} : (z', z'') \in \hat{\Omega}_t^*, z' \neq z'' \right\} > 0.$$

Suppose in addition that $\lim_{t \rightarrow \pm\infty} \left| \int_0^t \hat{\varrho}(s; r, d) \, ds \right| = \infty$ and if $d = 1$, then also

$$\liminf_{t \rightarrow \pm\infty} \frac{\ln \left(1 + \sqrt{\vartheta(t; r)} \right)}{2 \left| \int_0^t \hat{\varrho}(s; r, d) \, ds \right|} < 1.$$

Then the system with Hamiltonian (36) cannot have two different global solutions $(q_j(t), p_j(t))$, $t \in \mathbb{R}$, $j=1,2$, such that $(t, q_j(t), p_j(t)) \in \hat{\Omega}$ for all $t \in \mathbb{R}$, $j = 1, 2$.

Proof. In order to apply Theorem 2 in the case of Hamiltonian system

$$\dot{z} = JH'_z(t, z),$$

introduce the function $\hat{U}(z', z'') := \langle q' - q'', p' - p'' \rangle$. After the substitutions $p' = \frac{\partial L(t, u)}{\partial \dot{q}'}$, $p'' = \frac{\partial L(t, v)}{\partial \dot{q}''}$, this function coincides with the function $U(t, u, v)$ which appears when proving the Theorem 6. Hence, the estimate (35) implies that

$$\left| \hat{U}(z', z'') \right| \leq 2\omega^*(r) \left[1 + \sqrt{\vartheta(t; r)} \right]$$

once $\hat{V}(t, z', z'') \leq r$, and the inequality

$$|\hat{U}(z', z'')| \leq \frac{1}{2} \|z' - z''\|^2$$

together with definition of $\hat{q}(t; r, d)$ yields

$$\begin{aligned} \dot{\hat{U}}(z', z'')_{(JH'_z, JH''_z)} &= \left\langle I \left(\frac{\partial H(t, z')}{\partial z'} - \frac{\partial H(t, z'')}{\partial z''} \right), z' - z'' \right\rangle \geq \\ &\geq \hat{q}(t; r, d) \|z' - z''\|^{2d} \geq 2^d \hat{q}(t; r, d) |\hat{U}(z', z'')|^d \end{aligned}$$

if $\hat{V}(t, z', z'') \leq r$. The rest of the proof is based on the same arguments as the proof of previous theorem. □

As a corollary of Theorems 5, 7 we can get new sufficient conditions for the existence of almost periodic solutions to Lagrangian systems. Namely, consider the case where the following assumption is valid:

- (ϵ) the mappings $\mathcal{A}(\cdot, q) : \mathbb{R} \mapsto \text{Hom}(\mathbb{R}^m)$, $a(\cdot, q) : \mathbb{R} \mapsto \mathbb{R}^m$, $\Phi(\cdot, q) : \mathbb{R} \mapsto \mathbb{R}$ together with their first order partial derivatives in q are almost periodic uniformly for $q \in \mathbb{R}^m$ and the function $\Psi_*(q) := \inf_{t \in \mathbb{R}} \Psi(t, q)$ is coercive.

Denote

$$\Lambda^*(q) := \sup_{t \in \mathbb{R}} \Lambda(t, q), \quad \alpha_*(q) := \sup_{t \in \mathbb{R}} \|a(t, q)\|.$$

Since

$$H(t, z) + \Phi(t, q) + \Psi(t, q) \geq \frac{1}{2\Lambda^*(q)} (\|p\| - \alpha_*(q))^2 + \Psi_*(q),$$

and the function in the right-hand side of this inequality is coercive, then for any $r > 0$ the set

$$\begin{aligned} &\mathcal{V}(r) := \\ &= \text{cls} \bigcup_{t \in \mathbb{R}} \{ (p, q) \in \mathbb{R}^{2m} : H(t, p, q) + \Phi(t, q) + \Psi(t, q) \leq r, \omega_0 \leq \langle p, q \rangle \leq \omega^0 \} \end{aligned}$$

is compact (see (27), (28) for definitions of ω_0, ω^0).

Theorem 8. *Let the assumptions $(\alpha) - (\epsilon)$ be valid. Put*

$$r := \mathfrak{f}_{\theta,R} \left(\frac{K}{2\kappa} (\omega^0 - \omega_0) \right)$$

(the function $\mathfrak{f}_{\theta,R}(\cdot)$ is defined in Theorem 5) and suppose that there exist numbers $\varrho_* > 0$ and $d \geq 1$ such that

$$\left\langle I \left(\frac{\partial H(t, z')}{\partial z'} - \frac{\partial H(t, z'')}{\partial z''} \right), z' - z'' \right\rangle \geq \varrho_* \|z' - z''\|^{2d}$$

for all $(t, z', z'') \in \mathbb{R} \times \mathcal{V}(r) \times \mathcal{V}(r)$. Then the set $\mathcal{V}(r)$ contains one and only one global solution of the system with Hamiltonian $H(t, z)$, and this solution is almost periodic.

Proof. By Theorem 5 for any $s \in \mathbb{R}$, the Hamiltonian system

$$\dot{z} = JH'_z(t + s, z) \tag{37}$$

has a global solution taking values in $\mathcal{V}(r)$. Moreover, the same reasoning as in the proof of Theorem 7 shows that the set $\mathcal{V}(r)$ contains no other global solutions of system (37). Now to complete the proof, it remains only to apply the Amerio theorem (see, e.g., [37]). \square

Observe now that under the conditions imposed on $L(\cdot)$ the Hamiltonian belongs to $C^2(\mathbb{R}^{1+2m} \mapsto \mathbb{R})$. If we denote by $H''_{qq}(t, z)$ the partial Hesse matrix $\left\{ \frac{\partial H(t, q, p)}{\partial q_i \partial q_j} \right\}_{i, j=1}^m$, then it is easily seen that

$$\begin{aligned} & \left\langle I \left(\frac{\partial H(t, z')}{\partial z'} - \frac{\partial H(t, z'')}{\partial z''} \right), z' - z'' \right\rangle = \\ & = \left\langle \left[\int_0^1 \mathcal{A}^{-1}(t, sq' + (1-s)q'') ds \right] (p' - p''), p' - p'' \right\rangle - \\ & - \left\langle \left[\int_0^1 H''_{qq}(t, sz' + (1-s)z'') ds \right] (q' - q''), q' - q'' \right\rangle. \end{aligned}$$

Since the first summand of the right-hand side of this equality is positive definite quadratic form with respect to $p' - p''$ we arrive at conclusion that for the case where $d = 1$, in order that the function $\hat{\varrho}(t; r, d)$ be well-defined and positive, it is necessary that

$$\begin{aligned} \min_{\|\eta\|=1} \left\{ -\frac{1}{2} \left[\frac{\partial^2 \langle \mathcal{A}^{-1}(t, q)p, p \rangle}{\partial q_i \partial q_j} \eta_i \eta_j \right] + \left[\frac{\partial^2 \langle \mathcal{A}^{-1}(t, q)a(t, q), p \rangle}{\partial q_i \partial q_j} \eta_i \eta_j \right] - \right. \\ \left. - \frac{1}{2} \frac{\partial^2 \langle \mathcal{A}^{-1}(t, q)a(t, q), a(t, q) \rangle}{\partial q_i \partial q_j} \eta_i \eta_j + \frac{\partial^2 \Phi(t, q)}{\partial q_i \partial q_j} \eta_i \eta_j \right\} > 0 \tag{38} \end{aligned}$$

for all $(t, q, p) \in \hat{\Omega}$, and it is sufficient that the last inequality holds for all (t, q, p) such that $t \in \mathbb{R}$ and (p, q) belongs to the convex hull of the set $\hat{\Omega}_t$. Observe that the last set is contained in the convex hull of the set $\hat{V}^{-1}([0, r])$. If we treat the left hand side of the inequality (38) as a quadratic polynomial with respect to $u = \|p\|$, then we arrive at the following result.

Lemma 7. *Put*

$$\begin{aligned} \alpha_1(t, q) &:= 1/(2\Lambda(t, q)), & \beta_1(t, q) &:= \|\mathcal{A}^{-1}(t, q)a(t, q)\|, \\ \gamma_1(t, q) &:= \langle \mathcal{A}^{-1}(t, q)a(t, q), a(t, q) \rangle + \Psi(t, q), \\ \alpha_2(t, q) &:= \max_{\|y\|=1, \|\eta\|=1} \frac{\partial^2 \langle \mathcal{A}^{-1}(t, q)y, y \rangle}{\partial q_i \partial q_j} \eta_i \eta_j, \\ \beta_2(t, q) &:= \max_{\|\eta\|=1} \left\| \frac{\partial^2 \mathcal{A}^{-1}(t, q)a(t, q)}{\partial q_i \partial q_j} \eta_i \eta_j \right\|, \\ \gamma_2(t, q) &:= \min_{\|\eta\|=1} \left[2 \frac{\partial^2 \Phi(t, q)}{\partial q_i \partial q_j} - \frac{\partial^2 \langle \mathcal{A}^{-1}(t, q)a(t, q), a(t, q) \rangle}{\partial q_i \partial q_j} \right] \eta_i \eta_j, \end{aligned}$$

and suppose that for any $t \in \mathbb{R}$ the function $\Psi(t, \cdot) : \mathbb{R}^m \mapsto \mathbb{R}$ is quasiconvex and that there exists $r > 0$ such that for all $(t, q) \in \Psi^{-1}([0, r])$ the inequalities

$$\alpha_1(t, q)u^2 - 2\beta_1(t, q)u + \gamma_1(t, q) \leq r, \quad u \geq 0,$$

yield the inequality

$$\alpha_2(t, q)u^2 + 2\beta_2(t, q)u - \gamma_2(t, q) < 0.$$

Then the inequality (38) is valid for all (t, q, p) such that $t \in \mathbb{R}$ and (q, p) belongs to convex hull of the set $\hat{V}([0, r])$.

Remark 7. Since for any fixed t and $y \in \mathbb{R}^m$ the function $\langle \mathcal{A}^{-1}(t, \cdot)y, y \rangle$ is positive, it cannot be globally strictly concave. And if $\alpha_2(t, q) > 0$ at some point (t, q) , then the inequality (38) fails for all p with sufficiently large norm.

Example 2. Consider a Lagrangian system which describes motion of a particle constrained to move on time-varying helicoid under the impact of force of gravity and repelling potential field of force. The vibrating helicoid is given in 3-D space by the equations

$$\mathbf{r} = (q_1 \cos q_2, q_1 \sin q_2, \chi(t)q_2), \quad (q_1, q_2) \in \mathbb{R}^2,$$

where $\chi(\cdot) \in C^3(\mathbb{R} \mapsto (0, \infty))$ is a given function. Suppose that the function of repelling potential field is $\Pi(\mathbf{r}) = -k(\|\mathbf{r}\|^2 + \|\mathbf{r}\|^4)$, where $k \geq 1$ is a parameter.

Having assumed for simplicity the mass of particle and the acceleration of gravity to be unities, we get the following expression for kinetic energy

$$\frac{1}{2} \|\dot{\mathbf{r}}\|^2 = \frac{1}{2} (\dot{q}_1^2 + (\chi^2(t) + q_1^2) \dot{q}_2^2) + \chi(t) \dot{\chi}(t) q_2 \dot{q}_2 + \frac{1}{2} \dot{\chi}^2(t) q_2^2.$$

Since the term $\chi(t) \dot{\chi}(t) q_2 \dot{q}_2 + \frac{1}{2} \dot{\chi}^2(t) q_2^2$ gives the same contribution into the equations of motion as the term $-\frac{1}{2} \chi(t) \ddot{\chi}(t) q_2^2$, we obtain the following Lagrangian

$$L(t, q, \dot{q}) = \frac{1}{2} (\dot{q}_1^2 + (\chi^2(t) + q_1^2) \dot{q}_2^2) - \chi(t) q_2 - \frac{1}{2} \chi(t) \ddot{\chi}(t) q_2^2 + k [q_1^2 + \chi^2(t) q_2^2 + (q_1^2 + \chi^2(t) q_2^2)^2].$$

Hence, in this case $a(t, q) = 0$,

$$\begin{aligned} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle &= \dot{q}_1^2 + (\chi^2(t) + q_1^2) \dot{q}_2^2, \\ \Phi(t, q) &= k [q_1^2 + \xi(t) q_2^2 + (q_1^2 + \chi^2(t) q_2^2)^2] - \chi(t) q_2, \end{aligned}$$

where $\xi(t) := \chi^2(t) - \frac{1}{2k} \chi(t) \ddot{\chi}(t)$.

We suppose that the function $\chi(t)$ satisfies the following conditions:

$$\inf_{t \in \mathbb{R}} \chi(t) =: \chi_* \geq 1, \quad \sup_{t \in \mathbb{R}} \left| \frac{1}{\chi(t)} \frac{d^i \chi(t)}{dt^i} \right| =: \eta_i^* < \infty, \quad i = 1, 2, 3, \quad \eta_2^* \leq k.$$

Obviously that in this case $\xi(t) > \chi^2(t)/2$.

Put

$$\Psi(t, q) = k [q_1^2 + \xi(t) q_2^2 + 2 (q_1^2 + \chi^2(t) q_2^2)^2].$$

Then

$$\begin{aligned} \frac{\partial L}{\partial q_i} q_i + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i &= \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + q_1^2 \dot{q}_2^2 + \\ &+ 2k [q_1^2 + \xi(t) q_2^2 + 2 (q_1^2 + \chi^2(t) q_2^2)^2] - \chi(t) q_2 \geq \\ &\geq \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + 2\Psi(t, q) - \sqrt[4]{\frac{1}{4k}} \sqrt[4]{\langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + 2\Psi(t, q)} \geq \\ &\geq \kappa(R, k) \left(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \right) \end{aligned}$$

if $(\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q)) \geq R$, where $\kappa(R, k) := 2 - R^{-3/4} (2k)^{-1/4}$. Hence, the assumption (α) holds for arbitrary $R \geq (32k)^{-1/3}$.

Since $\langle \mathcal{A}(t, q)q, q \rangle = q_1^2 + (\chi^2(t) + q_1^2) q_2^2$, then the assumption (β) is valid with appropriately chosen function $\Theta(\cdot)$ and with $\bar{\Theta}(\Psi) = 2\Psi/k$.

Now let us verify the assumption (γ) . We have

$$\begin{aligned} \left| \frac{1}{2} \left\langle \frac{\partial \mathcal{A}(t, q)}{\partial t} \dot{q}, \dot{q} \right\rangle \right| &\leq \sup_{t \in \mathbb{R}} \left| \frac{\dot{\chi}(t)}{\chi(t)} \right| \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle = \eta_1^* \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle, \\ &\left| \frac{\partial(\Phi(t, q) + \Psi(t, q))}{\partial q_i} \dot{q}_i \right| \leq \sqrt{\langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle} \times \\ &\times \left[4k \sqrt{q_1^2 + \left[\frac{\xi(t)}{\chi(t)} \right]^2} q_2^2 + 12k (q_1^2 + \chi^2(t) q_2^2)^{3/2} + 1 \right] \leq \\ &\leq \sqrt{\langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle} \times \\ &\times \left(4k \sup_{t \in \mathbb{R}} \max \left\{ 1, \frac{\xi(t)}{\chi^2(t)} \right\} \sqrt{q_1^2 + \chi^2(t) q_2^2} + 12k (q_1^2 + \chi^2(t) q_2^2)^{3/2} + 1 \right) \leq \\ &\leq \sqrt{\langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle} \times \\ &\times \left[\left(6(2k)^{1/4} + (2k)^{3/4} (2 + \eta_2^*) \right) \Psi^{3/4}(t, q) + (2k)^{3/4} (2 + \eta_2^*) + 1 \right], \\ \left| \frac{\partial \Psi(t, q)}{\partial t} \right| &\leq \sup_{t \in \mathbb{R}} \max \left\{ \left| \frac{\dot{\xi}(t)}{\xi(t)} \right|, 4 \left| \frac{\dot{\chi}(t)}{\chi(t)} \right| \right\} \Psi(t, q) \leq (5\eta_1^* + \eta_3^*) \Psi(t, q). \end{aligned}$$

The same arguments as in the proof of Lemma 6 allows us to assert that for $\frac{1}{2} \langle \mathcal{A}(t, q) \dot{q}, \dot{q} \rangle + \Psi(t, q) \geq R$ the assumption (γ) is valid with $\theta = 1/4$ and

$$\begin{aligned} K = K(R, k) &:= \sqrt{2} \left[6 \cdot (2k)^{1/4} + 2(2k)^{3/4} (2 + \eta_2^*) \right] + \\ &+ R^{-1/4} (5\eta_1^* + \eta_3^*) + R^{-3/4} \sqrt{2} \left(2(2k)^{3/4} (2 + \eta_2^*) + 1 \right). \end{aligned}$$

Finally, Lemma 4 yields

$$\omega^0 \leq \max_{s \in [0, R]} \sqrt{4(R-s)s/k} = R\sqrt{2/k}, \quad \omega_0 \geq -R\sqrt{2/k}.$$

Hence, by Theorem 5, there exists a global solution $q_*(t)$, $t \in \mathbb{R}$, satisfying the inequality

$$\frac{1}{2} \langle \mathcal{A}(t, q_*(t)) \dot{q}_*(t), \dot{q}_*(t) \rangle + \Psi(t, q_*(t)) \leq r(k, R),$$

where

$$r(k, R) := \left[\frac{3\sqrt{2}RK(R, k)}{4\kappa(R, k)\sqrt{k}} + R^{3/4} \right]^{4/3}.$$

Observe, that if we put $R = (2k)^{-1/3}$, then $\kappa(k, (2k)^{-1/3}) = 1$ and $r(k, (2k)^{-1/3}) \leq Ck^{2/9}$, where a constant $C > 0$ does not depend on k . One can show that

$$K(k, (2k)^{-1/3}) \leq 3.78k + 1.59k^{3/4} + 11.78k^{1/4} + (5.3\eta_1^* + 1.1\eta_3^*)k^{1/12} + 1, 28\eta_2^* \leq (17.15 + 5.3\eta_1^* + 1.28\eta_2^* + 1.1\eta_3^*)k,$$

and

$$C \leq (15.28 + 4.47\eta_1^* + 2.28\eta_2^* + 0.93\eta_3^*)^{4/3}.$$

From this it follows that

$$(q_{1*}^2(t) + \chi^2(t)q_{2*}^2(t))^2 \leq Ck^{-7/9}/2 \quad \forall t \in \mathbb{R},$$

and

$$q_{1*}^2(t) + \chi^2(t)q_{2*}^2(t) \leq \sqrt{C/2}k^{-7/18} \quad \forall t \in \mathbb{R}.$$

Now consider the case where the function $\chi(\cdot)$ is almost periodic together with its derivatives up to the third order. In order to apply Lemma 7, observe that

$$\alpha_1(t, q) = \frac{1}{2(\chi^2(t) + q_1^2)}, \quad \alpha_2(t, q) = 2\frac{3q_1^2 - \chi^2(t)}{(\chi^2(t) + q_1^2)^3},$$

$$\beta_1(t, q) = \beta_2(t, q) = 0,$$

and it is no hard to show that in our case

$$\gamma_2(t, q) \geq 2k [\min\{1, \xi(t)\} + 2(q_1^2 + \chi^2(t)q_2^2)],$$

Now it is easily seen that the conditions of Lemma 7 will hold true if on the set where $\Psi(t, q) \leq r$ there holds the inequality

$$2\frac{3q_1^2 - \chi^2(t)}{(\chi^2(t) + q_1^2)^2}(r - \Psi(t, q)) \leq k [\min\{1, \xi(t)\} + 2(q_1^2 + \chi^2(t)q_2^2)].$$

Observe that $\sup_{u \geq 0} \frac{3u^2 - \chi^2(t)}{(\chi^2(t) + u^2)^2} = \frac{9}{16\chi^2(t)} \leq \frac{9}{16\chi_*^2}$ and $\xi(t) \geq \chi_*^2/2$. Thus, in the case where $r = r(k, (2k)^{-1/3})$, we get the following sufficient condition for almost periodicity of solution $q_*(t)$ in terms of restrictions on parameter k :

$$Ck^{2/9} \leq 4k \min\{1, \chi_*^2/2\}, \quad k \geq \max\{1, \eta_2^*\}$$

or

$$k \geq \max \left\{ 1, \eta_2^*, \left[\frac{C}{4 \min\{1, \chi_*^2/2\}} \right]^{9/7} \right\}.$$

6 Conclusions

The technique applied in this paper for studying essentially nonlinear nonautonomous systems by means of a pair of auxiliary functions allows us to generalize a number of earlier known results concerning the questions of existence and uniqueness of bounded, proper and almost periodic solutions. In the case where the estimating function is a quadratic form with varying matrix, the estimates obtained for V-bounded solutions can be efficiently applied to describe asymptotic behavior of solutions when $t \rightarrow \pm\infty$. For Lagrangian systems with certain directional quasiconvexity property, there exists a V-W pair which allows to establish sufficient conditions for existence of V-bounded solutions. Our approach yields uniqueness theorems for V-bounded solutions as well. As a consequence of that, we have obtained new sufficient conditions for the existence of almost periodic solutions to Lagrangian systems.

This work was partially supported by the Fundamental Research State Fund of Ukraine (Project 29.1/025).

- [1] *Lagoda V., Parasyuk I.* Ważewski topological principle and V-bounded solutions of nonlinear systems, arXiv:0901.0234v1 [math.CA]
- [2] *Ważewski T.* Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires // Ann. Soc. Polon. Math. – 1947. – **20**. – P. 279 – 313.
- [3] *Hartman P.* Ordinary differential equations. – New York-London-Sydney: John Wiley and Sons, 1964. – 612 pp.

- [4] *Conley C.* A new statement of Ważewski's theorem and an example, *Ordin. part. Differ. Equat., Proc. 4th Conf., Dundee, 1976.* – *Lect. Notes Math.*, **564**. – 1976. – P. 61 – 71.
- [5] *Conley C.* Isolated invariant sets and the Morse index, *Conf. Board Math. Sci.*, **38**. – Providence, RI: American Mathematical Society, 1978.
- [6] *Cheresiz V.M.* Stable and conditionally stable almost-periodic solutions of V-monotone systems // *Siberian Math. J.* – 1974. – **15**, no. 1. – P. 116 – 125.
- [7] *Conley C.*, An application of Ważewski's method to a non-linear boundary value problem which arises in population genetics // *J. Math. Biol.* – 1975. – **2**. – P. 241 – 249.
- [8] *Ivanov J.A.* Topological conditions for the existence of bounded solutions of quasihomogeneous systems // *J. Math. Sciences.* – 1987. – **37**, no. 3. – P. 1144 – 1149.
- [9] *Ivanov O.A.* Ważewski's topological principle and existence of bounded solutions of quasihomogeneous systems // *Vestn. Leningr. Univ., Mat. Mekh. Astron.* – 1985. – no. 1. – P. 109 – 110.
- [10] *Ortega R.* Retracts, fixed point index and differential equations // *R. Acscd. Cien. Serie A. Mat.* – 2008. – **102**. – P. 89 – 100.
- [11] *Krasnosel'skii M.A., Zabreiko P.P.* Geometrical Methods of Nonlinear Analysis. A Series of Comprehensive Studies in Mathematics. – Berlin-Heidelberg-New York-Tokio: Springer Verlag, 1984. – **263**. – 409 pp.
- [12] *Mawhin J., Ward James R. (Jr.)* Guiding-like functions for periodic or bounded solutions of ordinary differential equations // *Discr. Cont. Dyn. Sys.* – 2002. – **8**. – P. 39 – 54.
- [13] *Mawhin J., Thompson H.B.* Periodic or bounded solutions of Caratheodory systems of ordinary differential equations // *J. Dyn. Diff. Eq.* – 2003. – **15**, no. 2–3. – P. 327 – 334.
- [14] *Avramescu C.* Asymptotic behavior of solutions of nonlinear differential equations and generalized guiding functions // *Electron. J. Qualit. Theory Differ. Equat.* – 2003. – no. 13. – P. 1 – 9. (www.math.u-czeged.hu/ejqtde)
- [15] *Perov A.I., Trubnikov Ju.V.* Monotonic differential equations. I // *Diff. Eq.* – 1974. – **10**. – P. 619 – 627.
- [16] *Trubnikov Yu.V., Perov A.I.* Differential equations with monotone nonlinearities. – Minsk: Nauka i Tekhnika, 1986. – 200 pp. (in Russian).
- [17] *Mukhamadiev Eh., Nazmiddinov H., Sadovskij B.N.* Application of the Schauder-Tikhonov principle to the problem of bounded solutions of differential equations // *Funct. Anal. Appl.* – 1972. – **6**. – P. 246 – 247.

- [18] *Perov A.I.* On bounded solutions of nonlinear systems of ordinary differential equations // Vestnik Voronezh State Univ., Ser. Phys. Math. – 2003. – no. 1. – P. 165 – 168. (In Russian).
- [19] *Ward James R. (Jr.)* Homotopy and bounded solutions of ordinary differential equations // J. Diff. Eq. – 1994. – **107**, no. 2. – P. 428 – 445.
- [20] *Samoilenko A.M., Boichuk A.A., Boichuk An.A.* Solutions of weakly-perturbed linear systems bounded on the entire axis // Ukr. Math. J. – 2002. – **54**, no. 11. – P. 1842 – 1858.
- [21] *Blot J.* Calculus of variations in mean and convex Lagrangians // J. Math. Anal. Appl. – 1988. – **134**, no. 2. – P. 312 – 321.
- [22] *Berger M.S., Zhang L.* A new method for large quasiperiodic nonlinear oscillations with fixed frequencies for the nondissipative conservative systems // Comm. Appl. Nonlinear Anal. – 1995. – **2**. – P. 79 – 106.
- [23] *Blot J., Cieutat P., Mawhin J.* Almost-periodic oscillations of monotone second-order systems // Adv. Diff. Eq. – 1997. – **2**. – P. 693 – 714.
- [24] *Zakharin S.F., Parasyuk I.O.* Generalized and classical almost periodic solutions of Lagrangian systems // Funkcial. Ekvac. – 1999. – **42**. – P. 325 – 338.
- [25] *Cieutat P.* Almost periodic solutions of second-order systems with monotone fields on a compact subset // Nonlin. Anal.: Theory, Meth. Appl. – 2003. – **53**, no. 6. – P. 751 – 763.
- [26] *Cieutat P.* Bounded and almost periodic solutions of convex Lagrangian systems // J. Diff. Equat. – 2003. – **190**. – P. 108 – 130.
- [27] *Cheban D., Mammana C.* Invariant manifolds, almost periodic and almost automorphic solutions of second-order monotone equations // Internat. J. Evolut. Equat. – 2005. – **1**, no. 4. – P. 319 – 343.
- [28] *Herzog G.* On bounded solutions of second order systems // Demonstratio Math. – 2006. – **39**, no. 4. – P. 793 – 801.
- [29] *Agarwal R.P., O'Regan D., Saker S.* Properties of bounded solutions of nonlinear dynamic equations on time scales // Can. Appl. Math. Q. – 2006. – **14**, no. 1. – P. 1 – 10.
- [30] *Slyusarchuk V.E.* Necessary and sufficient conditions for the Lipschitz invertibility of the nonlinear differential operator $d/dt - f$ in the space of bounded functions on the real axis // Nonlinear Oscil. – 2001. – **4**, no. 2. – P. 272 – 277.
- [31] *Hrod I.M.* Conditions for the existence of bounded solutions of one class of nonlinear differential equations // Ukr. Math. J. – 2006. – **58**, no. 3. – P. 357 – 367.
- [32] *Karpinska W.* On bounded solutions of nonlinear first- and second-order equations with a Caratheodory function // J. Math. Anal. Appl. – 2007. – **334**, no. 2. – P. 1462 – 1480.

- [33] *Daleckiĭ Yu.L., Krein M.G.* Stability of solutions of differential equations in Banach space. Translations of Mathematical Monographs, **43**. – Providence, R.I.: American Mathematical Society, 1974. – 386 pp.
- [34] *Mitropolsky Yu.A., Samoilenko A.M., Kulik V.L.* Dichotomies and stability in nonautonomous linear systems. Stability and Control: Theory, Methods and Applications, **14**. – London: Taylor & Francis, 2003. – 368 pp.
- [35] *Samoilenko A.M.* On the exponential dichotomy on \mathbb{R} of linear differential equations in \mathbb{R}^n // Ukr. Math. J. – 2002. – **53**, no. 3. – P. 407 – 426.
- [36] *Gantmakher F.R.* Theory of matrices. – Moscow: Nauka, 1988. – 549 pp. (in Russian).
- [37] *Fink A.M.* Almost periodic differential equations. Lecture Notes in Mathematics, **377**. – Berlin-Heidelberg-New York: Springer-Verlag. – 1974. – VIII. – 336 pp.

ІСНУВАННЯ V-ОБМЕЖЕНИХ РОЗВ'ЯЗКІВ НЕАВТОНОМНИХ НЕЛІНІЙНИХ СИСТЕМ ТА ТОПОЛОГІЧНИЙ ПРИНЦИП ВАЖЕВСЬКОГО

Володимир ЛАГОДА, Ігор ПАРАСЮК

Київський національний університет імені Тараса Шевченка
вул. Володимирська 64, Київ 01601
email: pio@univ.kiev.ua

Встановлено низку нових достатніх умов існування глобальних (визначених на всій дійсній осі) розв'язків нелінійних неавтономних систем за допомогою топологічного принципу Важевського. Досліджувані системи характеризуються властивістю монотонності відносно деякої допоміжної напрямної функції $W(t, x)$, залежної від часу та фазових координат. Інша допоміжна функція $V(t, x)$ така, що $\lim_{\|x\| \rightarrow \infty} V(t, x) = \infty$ для всіх $t \in \mathbb{R}$, використовується для оцінювання розташування глобальних розв'язків у розширеному фазовому просторі. Розроблений підхід застосовується до лагранжевих систем, зокрема, для встановлення нових достатніх умов існування майже періодичних розв'язків.