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THE CLASSICAL ABRAHAM-LORENTZ ELECTRON MASS THEORY ANALYSIS. PART 2: THE SHELL MODEL REVISITING

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In Part 2 of the work I reanalyze the Feynman's proper time paradigm and the Lorentz force derivation from the classical Lagrangian and Hamiltonian points of view, and also construct its nontrivial relativistic generalization which is compatible with the vacuum field theory approach. As a consequence, the classical Abraham-Lorentz electromagnetic electron mass problems is argued and analyzed in detail.

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В Част.2 праці мною проаналізовані парадигма власного часу Фейнмана та вивід сили Лорентца з класичних Лагранжевої та Гамільтонової точок зору, а також побудова її нетривільного узагальнення сумісного з вакуумно-польовим підходом. Як наслідок, класична проблема Абрагама-Лорентца щодо маси електрона отримала свою аргументацію, а також більш детальний аналіз.

This paper is the second part of a longer project, whose first part is published in [52]. The numeration of the sections continues that of the paper [52].

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1. The modified Lorentz force, radiation theory and the Abraham–Lorentz electron inertia problem

1.1. Feynman proper time paradigm geometric analysis

We develop further the vacuum field theory approach within the Feynman proper time paradigm, devised before in [13, 11], to the electromagnetic Maxwell and Lorentz electron theories and show that they should be suitably modified: namely, the basic Lorentz force equations should be generalized following the Landau-Lifschitz least action recipe [39], taking also into account the pure electromagnetic field impact. When applied the devised vacuum field theory approach to the classical electron shell model, the resulting Lorentz force expression appears to satisfactorily explain the electron inertial mass term exactly coinciding with the electron relativistic mass, thus confirming the well known assumption [36, 58] by M. Abraham and H. Lorentz.

As was reported by F. Dyson [22, 23], the original Feynman approach derivation of the electromagnetic Maxwell equations was based on an *a priori* general form of the classical Newton type force, acting on a charged point particle moving in three-dimensional space $\mathbb{R}^3$ endowed with the canonical Poisson brackets on the phase variables, defined on the associated tangent space $T(\mathbb{R}^3)$. As a result of this approach, only the first part of the Maxwell equations was derived, as the second part, owing to F. Dyson [22], is related with the charged matter nature, which appeared to be hidden. Trying to complete this Feynman approach to the derivation of Maxwell's equations more systematically we have observed [11] that the original Feynman's calculations, based on Poisson brackets analysis, were performed on the *tangent space* $T(\mathbb{R}^3)$ which is, subject to the problem posed, not physically proper. The true Poisson brackets can be correctly defined only on the *coadjoint phase space* $T^*(\mathbb{R}^3)$, as seen from the classical Lagrangian equations and the related Legendre transformation [2, 3, 31, 8] from $T(\mathbb{R}^3)$ to $T^*(\mathbb{R}^3)$. Moreover, within this observation, the corresponding dynamical Lorentz type equation for a charged point particle should be written for the particle momentum, not for the particle velocity, whose value is well defined only with respect to the proper relativistic reference frame, associated with the charged point particle owing to the fact that the Maxwell equations are Lorentz invariant.

Thus, from the very beginning, we shall reanalyze the structure of the Lorentz force exerted on a moving charged point particle with a charge $\xi \in \mathbb{R}$ by another point charged particle with a charge $\xi_f \in \mathbb{R}$, making use of the classical Lagrangian approach, and rederive the corresponding electromagnetic Maxwell equations. The latter appears to be strongly related to the charged point mass structure of the electromagnetic origin as was suggested by R. Feynman and F. Dyson.

Consider a charged point particle moving in an electromagnetic field. For its description, it is convenient to introduce a trivial fiber bundle structure $\pi : \mathcal{M} \to \mathbb{R}^3$, $\mathcal{M} = \mathbb{R}^3 \times G$, with the abelian structure group $G := \mathbb{R}\backslash\{0\}$, equivariantly acting on the canonically symplectic coadjoint space $T^*(\mathcal{M})$ endowed both with the canonical symplectic structure

$$\omega^{(2)}(p,y;r,g)=dpr^{*}\alpha^{(1)}(r,g)=\langle dp,\wedge dr\rangle+\langle dy,\wedge g^{-1}dg\rangle_{\mathcal{G}}+\langle ydg^{-1},\wedge dg\rangle_{\mathcal{G}} \quad (1)$$

for all $(p,y;r,g)\in T^{*}(\mathcal{M})$, where $\alpha^{(1)}(r,g):=\langle p,dr\rangle+\langle y,g^{-1}dg\rangle_{\mathcal{G}}\in T^{*}(\mathcal{M})$ is the corresponding Liouville form on $\mathcal{M}$, and with a connection one-form $\mathcal{A}: M\to T^{*}(M)\times\mathcal{G}$ as $\mathcal{A}(r,g):=g^{-1}\langle\xi A(r),dr\rangle g+g^{-1}dg$, with $\xi\in\mathcal{G}^{*}$, $(r,g)\in\mathbb{R}^{3}\times G$, and $\langle\cdot,\cdot\rangle$ being the scalar product in $\mathbb{E}^{3}$. The corresponding curvature 2-form $\Sigma^{(2)}\in\Lambda^{2}(\mathbb{R}^{3})\otimes\mathcal{G}$ is

$$\Sigma^{(2)}(r):=d\mathcal{A}(r,g)+\mathcal{A}(r,g)\wedge\mathcal{A}(r,g)=\xi\sum_{i,j=1}^{3}F_{ij}(r)dr^{i}\wedge dr^{j},$$

where

$$F_{ij}(r):=\frac{\partial A_{j}}{\partial r^{i}}-\frac{\partial A_{i}}{\partial r^{j}} \quad (2)$$

for $i,j=\overline{1,3}$ is the electromagnetic tensor with respect to the reference frame $\mathcal{K}_{t}$, characterized by the phase space coordinates $(r,p)\in T^{*}(\mathbb{R}^{3})$. As an element $\xi\in\mathcal{G}^{*}$ is still not fixed, it is natural to apply the standard [2, 3, 8] invariant Marsden–Weinstein–Meyer reduction to the orbit factor space $\tilde{P}_{\xi}:=P_{\xi}/G_{\xi}$ subject to the related momentum mapping $l: T^{*}(\mathcal{M})\to\mathcal{G}^{*}$, constructed with respect to the canonical symplectic structure (1) on $T^{*}(\mathcal{M})$, where, by definition, $\xi\in\mathcal{G}^{*}$ is constant, $P_{\xi}:=l^{-1}(\xi)\subset T^{*}(\mathcal{M})$ and $G_{\xi}=\{g\in G: Ad_{G}^{*}\xi\}$ is the isotropy group of the element $\xi\in\mathcal{G}^{*}$.

As a result of the Marsden–Weinstein–Meyer reduction, one finds that $G_{\xi}\simeq G$, the factor-space $\tilde{P}_{\xi}\simeq T^{*}(\mathbb{R}^{3})$ is endowed with a suitably reduced symplectic structure $\bar{\omega}_{\xi}^{(2)}\in T^{*}(\tilde{P}_{\xi})$ and the corresponding Poisson brackets on the reduced manifold $\tilde{P}_{\xi}$ are $\{r^{i},r^{j}\}_{\xi}=0$, $\{p_{j},r^{i}\}_{\xi}=\delta_{j}^{i}$, $\{p_{i},p_{j}\}_{\xi}=\xi F_{ij}(r)$ for $i,j=\overline{1,3}$, considered with respect to the reference frame $\mathcal{K}_{t}$. Introducing a new momentum variable

$$\tilde{\pi}:=p+\xi A(r) \quad (3)$$

on $\tilde{P}_{\xi}$, it is easy to verify that $\bar{\omega}_{\xi}^{(2)}\to\tilde{\omega}_{\xi}^{(2)}:=\langle d\tilde{\pi},\wedge dr\rangle$, giving rise to the following "*minimal interaction*" canonical Poisson brackets:

$$\{r^{i},r^{j}\}_{\tilde{\omega}_{\xi}^{(2)}}=0,\ \{\tilde{\pi}_{j},r^{i}\}_{\tilde{\omega}_{\xi}^{(2)}}=\delta_{j}^{i},\ \{\tilde{\pi}_{i},\tilde{\pi}_{j}\}_{\tilde{\omega}_{\xi}^{(2)}}=0$$

for $i,j=\overline{1,3}$ with respect to some new reference frame $\tilde{\mathcal{K}}_{t'}$, characterized by the phase space coordinates $(r,\tilde{\pi})\in\tilde{P}_{\xi}$ and a new evolution parameter $t'\in\mathbb{R}$ if and only if the Maxwell field compatibility equations

$$\frac{\partial F_{ij}}{\partial r_{k}}+\frac{\partial F_{jk}}{\partial r_{i}}+\frac{\partial F_{ki}}{\partial r_{j}}=0$$

are satisfied on $\mathbb{R}^3$ for all $i, j, k = \overline{1,3}$ with the curvature tensor (2).

Now we proceed to a dynamic description of the interaction between two moving charged point particles ξ and ξ_f, moving respectively, with the velocities $u := dr/dt$ and $u_f := dr_f/dt$ subject to the reference frame $\mathcal{K}_t$. Unfortunately, there is a fundamental problem in correct formulation of a physically suitable action functional and the related least action condition. There are clearly possibilities such as

$$S_p^{(t)} := \int_{t_1}^{t_2} dt \mathcal{L}_p^{(t)}[r; dr/dt] \tag{4}$$

on a temporal interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference frame $\mathcal{K}_t$,

$$S_p^{(t')} := \int_{t_1'}^{t_2'} dt' \mathcal{L}_p^{(t')}[r; dr/dt'] \tag{5}$$

on a temporal interval $[t_1', t_2'] \subset \mathbb{R}$ with respect to the moving reference frame $\mathcal{K}_{t'}$ and

$$S_p^{(\tau)} := \int_{\tau_1}^{\tau_2} d\tau \mathcal{L}_p^{(\tau)}[r; dr/d\tau] \tag{6}$$

on a temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the proper time reference frame $\mathcal{K}_\tau$, naturally related to the moving charged point particle ξ.

It was first observed by Poincaré and Minkowski [49] that the temporal differential $d\tau$ is not a closed differential one-form, which physically means that a particle can traverse many different paths in space $\mathbb{R}^3$ with respect to the reference frame $\mathcal{K}_t$ during any given proper time interval $d\tau$, *naturally* related to its motion. This fact was stressed [24, 44, 49, 50] by Einstein,Minkowski and Poincaré, and later exhaustively analyzed by R. Feynman, who argued [26] that the dynamical equation of a moving point charged particle is physically sensible only with respect to its proper time reference frame. This is Feynman's proper time reference frame paradigm, which was recently further elaborated and applied both to the electromagnetic Maxwell equations in [29, 28, 30] and to the Lorentz type equation for a moving charged point particle under external electromagnetic field in [11, 13, 9, 8]. As it was there argued from a physical point of view, the least action principle should be applied only to the expression (6) written with respect to the proper time reference frame $\mathcal{K}_\tau$, whose temporal parameter $\tau \in \mathbb{R}$ is independent of an observer and is a closed differential one-form. Consequently, this action functional is also mathematically sensible, which in part reflects the Poincaré's and Minkowski's observation that the infinitesimal quadratic interval $d\tau^2 = (dt')^2 - |dr - dr_f|^2$, relating the reference frames $\mathcal{K}_{t'}$ and $\mathcal{K}_\tau$, can be invariantly used for the four-dimensional relativistic geometry. The most natural way to contend with this problem is to first consider the quasi-relativistic dynamics of the charged point particle ξ with respect to the moving reference frame $\mathcal{K}_{t'}$ subject to which the charged point particle ξ_f is at rest. Therefore, it possible to write down a suitable action functional (5), up to $O(1/c^4)$, as the light velocity $c \to \infty$, where the quasi-classical Lagrangian function $\mathcal{L}_p^{(t')}[r; dr/dt']$ can be naturally chosen as

$$\mathcal{L}_p^{(t')}[r; dr/dt'] := \frac{1}{2}m'(r)|dr/dt' - dr_f/dt'|^2 - \xi\varphi'(r). \tag{7}$$

where $m'(r) \in \mathbb{R}_+$ is the charged particle ξ inertial mass parameter and $\varphi'(r)$ is the potential function generated by the charged particle ξ_f at a point $r \in \mathbb{R}^3$ with respect to the reference frame $\mathcal{K}_{t'}$. Since the standard temporal relationships between reference frames $\mathcal{K}_t$ and $\mathcal{K}_{t'}$:

$$dt' = dt\sqrt{1 - |dr_f/dt'|^2}, \tag{8}$$

as well as between the reference frames $\mathcal{K}_{t'}$ and $\mathcal{K}_\tau$:

$$d\tau = dt'\sqrt{1 - |dr/dt' - dr_f/dt'|^2}, \tag{9}$$

give rise, up to $O(1/c^2)$, as $c \to \infty$, to $dt' \simeq dt$ and $d\tau \simeq dt'$, respectively, it is easy to verify that the least action condition $\delta S_p^{(t')} = 0$ is equivalent to the dynamical equation

$$d\pi/dt = \nabla\mathcal{L}_p^{(t')}[r; dr/dt] = (\frac{1}{2}\left|dr/dt - dr_f/dt\right|^2)\nabla m - \xi\nabla\varphi(r),$$

where we have defined the generalized canonical momentum as

$$\pi := \partial\mathcal{L}_p^{(t')}[r; dr/dt]/\partial(dr/dt) = m(dr/dt - dr_f/dt), \tag{10}$$

with the dash signs dropped and denoted by "∇" the usual gradient operator in $\mathbb{E}^3$. Equating the canonical momentum expression 10) with respect to the reference frame $\mathcal{K}_{t'}$ to that of (3) with respect to the canonical reference frame $\tilde{\mathcal{K}}_{t'}$, and identifying the reference frame $\tilde{\mathcal{K}}_{t'}$ with $\mathcal{K}_{t'}$, one obtains that

$$m(dr/dt - dr_f/dt) = mdr/dt - \xi A(r),$$

giving rise to the important inertial particle mass determining expression

$$m = -\xi\varphi(r), \tag{11}$$

which right away follows from the relationship

$$\varphi(r)dr_f/dt = A(r). \tag{12}$$

The latter is well known in the classical electromagnetic theory [36, 39] for potentials $(\varphi, A) \in T^*(M^4)$ satisfying the Lorentz condition

$$\partial\varphi(r)/\partial t + \langle\nabla, A(r)\rangle = 0, \tag{13}$$

yet the expression (11) looks very nontrivial in relating the "*inertial*" mass of the charged point particle ξ to the electric potential, being both generated by the ambient charged point particles ξ_f. As was argued in articles [9, 11, 55], the above mass phenomenon is closely related and from a physical perspective shows its deep relationship to the classical electromagnetic mass problem.

Before further analysis of the completely relativistic the charge ξ motion under consideration, we substitute the mass expression (11) into the quasi-relativistic action functional (5) with the Lagrangian (7). As a result, we obtain two possible action functional expressions, taking into account two main temporal parameters choices:

$$S_p^{(t')} = -\int_{t_1'}^{t_2'} \xi\varphi'(r)(1 + \tfrac{1}{2}|dr/dt' - dr_f/dt'|^2)dt'$$

on an interval $[t_1', t_2'] \subset \mathbb{R}$, or

$$S_p^{(\tau)} = -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 + \tfrac{1}{2}|dr/d\tau - dr_f/d\tau|^2)d\tau \tag{14}$$

on an $[\tau_1, \tau_2] \subset \mathbb{R}$. The direct relativistic transformations of (14) entail that

$$\begin{aligned} S_p^{(\tau)} &= -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 + \tfrac{1}{2}\left|dr/d\tau - dr_f/d\tau\right|^2)d\tau \simeq \\ &\simeq -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 + \left|dr/d\tau - dr_f/d\tau\right|^2)^{1/2}d\tau = \\ &= -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 - |dr/dt' - dr_f/dt'|)^{-1/2}d\tau = -\int_{t_1'}^{t_2'} \xi\varphi'(r)dt', \end{aligned}$$

giving rise to the correct, from the physical point of view, relativistic action functional form (5), suitably transformed to the proper time reference frame representation (6) via the Feynman proper time paradigm. Thus, we have shown that the true action functional procedure consists in a physically motivated choice of either the action functional expression form (4) or (5). Then, it is transformed to the proper time action functional representation form (6) within the Feynman paradigm, and the least action principle is applied.

1.2. The least action functionals analysis

Concerning the above discussed problem of describing the motion of a charged point particle ξ in the electromagnetic field generated by another moving charged point particle ξ_f, it must be mentioned that we have chosen the quasi-relativistic functional expression (7) in the form (5) with respect to the moving reference frame $\mathcal{K}_{t'}$, because its form is physically reasonable and acceptable, since the charged point particle ξ_f is then at rest, generating no magnetic field.

Based on the above relativistic action functional expression

$$S_p^{(\tau)} := -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)\sqrt{1 + |dr/d\tau - dr_f/d\tau|^2}d\tau$$

written with respect to the proper reference from $\mathcal{K}_\tau$, one finds the following evolution equation:

$$d\pi_p/d\tau = -\xi\nabla\varphi'(r)\sqrt{1 + |dr/d\tau - dr_f/d\tau|^2}, \tag{15}$$

where the generalized momentum is given exactly by the relationship (10): $\pi_p = m(dr/dt - dr_f/dt)$. Making use of the relativistic transformation (8) and the next one (9), the equation (15) is easily transformed to

$$\frac{d}{dt}(p + \xi A) = -\nabla\varphi(r)(1 - \left|u_f\right|^2), \tag{16}$$

where we took into account the related definitions: (11) for the charged particle ξ mass, (12) for the magnetic vector potential and $\varphi(r) = \varphi'(r)/\sqrt{1 - |u_f|^2}$ for the scalar electric potential with respect to the laboratory reference frame $\mathcal{K}_t$. Equation (16) can be further transformed, using elementary vector algebra, to the classical Lorentz type form:

$$\frac{dp}{dt} = \xi E + \xi u \times B - \xi\nabla\langle u - u_f, A\rangle, \tag{17}$$

where

$$E := -\partial A/\partial t - \nabla\varphi \tag{18}$$

is the related electric field and

$$B := \nabla \times A \tag{19}$$

is the related magnetic field, exerted by the moving charged point particle ξ_f on the charged point particle ξ with respect to the laboratory reference frame $\mathcal{K}_t$. The Lorentz type force equation (17) was obtained in [11] in terms of the moving reference frame $\mathcal{K}_{t'}$, and recently reanalyzed in [9, 51]. The obtained results follow in part [59] from Ampère's classical works on constructing the magnetic force between two neutral conductors with stationary currents.

2. Analysis of the Maxwell and Lorentz force equations

2.1. The damping Abraham-Lorentz reaction force: the least action functional approach

As a moving charged particle ξ_f generates the suitable electric field (18) and magnetic field (19) via their electromagnetic potential $(\varphi, A) \in T^*(M^4)$ with respect to a laboratory reference frame $\mathcal{K}_t$, we will supplement them naturally by means of the external material equations describing the relativistic charge conservation law:

$$\partial\rho/\partial t + \langle \nabla, J \rangle = 0, \tag{20}$$

where $(\rho, J) \in T^*(M^4)$ is a related four-vector for the charge and current distribution in the space $\mathbb{R}^3$. Moreover, one can augment the equation (20) with the experimentally well established Gauss law

$$\langle \nabla, E \rangle = \rho \tag{21}$$

to calculate the quantity $\Delta\varphi := \langle \nabla, \nabla\varphi \rangle$ from the expression (18):

$$\Delta\varphi = -\frac{\partial}{\partial t}\langle \nabla, A \rangle - \langle \nabla, E \rangle.$$

Having taken into account the relativistic Lorentz condition (13) and the expression (21) one easily finds that the wave equation

$$\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = \rho \tag{22}$$

holds with respect to the laboratory reference frame $\mathcal{K}_t$. Applying the rot-operation "$\nabla\times$" to the expression (18) we obtain, owing to the expression (19), the equation

$$\nabla \times E + \frac{\partial B}{\partial t} = 0, \tag{23}$$

giving rise, together with (21), to the first pair of the classical Maxwell equations. To obtain the second pair of the Maxwell equations, it is first necessary to apply the rot-operation "$\nabla\times$"to the expression (19):

$$\nabla \times B = \frac{\partial E}{\partial t} + \Big(\frac{\partial^2 A}{\partial t^2} - \Delta A\Big) \tag{24}$$

and then apply $-\frac{\partial}{\partial t}$ to the wave equation (22) to obtain

$$\begin{aligned} -\tfrac{\partial^2}{\partial t^2}(\tfrac{\partial\varphi}{\partial t}) + \langle \nabla, \nabla\tfrac{\partial\varphi}{\partial t} \rangle = \tfrac{\partial^2}{\partial t^2}\langle \nabla, A \rangle - \big\langle \nabla, \nabla\langle \nabla, A \rangle\big\rangle = \\ = \langle \nabla, \tfrac{\partial^2 A}{\partial t^2} - \nabla \times (\nabla \times A) - \Delta A \rangle = \langle \nabla, \tfrac{\partial^2 A}{\partial t^2} - \Delta A \rangle = \langle \nabla, J \rangle. \end{aligned} \tag{25}$$

The result (25) leads to the relationship

$$\frac{\partial^2 A}{\partial t^2} - \Delta A = J, \tag{26}$$

if we take into account that both the vector potential $A \in \mathbb{E}^3$ and the vector of current $J \in \mathbb{E}^3$ are determined up to a rot-vector expression $\nabla \times S$ for some smooth vector-function $S : M^4 \to \mathbb{E}^3$. Inserting the relationship (26) into (24), we obtain 23) and the second pair of the Maxwell equations:

$$\nabla \times B = \frac{\partial E}{\partial t} + J, \ \ \nabla \times E = \frac{\partial B}{\partial t}. \tag{27}$$

It is important that the system of equations (27) can be represented by means of the least action principle $\delta S^{(t)}_{f-p} = 0$, where the action functional

$$S^{(t)}_{f-p} := \int_{t_1}^{t_2} dt \mathcal{L}^{(t)}_{f-p}$$

is defined on an interval $[t_1, t_2] \subset \mathbb{R}$ by the Landau-Lifschitz type [39] Lagrangian function

$$\mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3 r \big((|E|^2 - |B|^2)/2 + \langle J, A\rangle - \rho\varphi \big) \tag{28}$$

with respect to the laboratory reference frame $\mathcal{K}_t$, which is subject to the electromagnetic field a unique and physically reasonable. From (28) we deduce that the generalized field momentum

$$\pi_f := \partial \mathcal{L}^{(t)}_{f-p} / \partial(\partial A/\partial t) = -E$$

and its evolution is given as

$$\partial \pi_f / \partial t := \delta \mathcal{L}^{(t)}_{f-p} / \delta A = J - \nabla \times B,$$

which is equivalent to the first Maxwell equation of (27). As the Maxwell equations allow the least action representation, it is easy to derive [2, 3, 8, 9, 55] their dual Hamiltonian formulation with the Hamiltonian function

$$H_{f-p} := \int_{\mathbb{R}^3} d^3 r \langle \pi_f, \partial A/\partial t\rangle - \mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3 r \big((|E|^2 + |B|^2)/2 - \langle J, A\rangle \big),$$

satisfying the invariant condition $dH_{f-p}/dt = 0$ for all $t \in \mathbb{R}$.

It is worth noting here that the Maxwell equations were derived under the important condition that the charged system $(\rho, J) \in T(M^4)$ exerts no influence on the ambient electromagnetic field potentials $(\varphi, A) \in T^*(M^4)$. As this is not actually the case owing to the damping radiation reaction on accelerated charged particles, one can try to describe this self-interacting influence by means of the modified least action principle, making use of the Lagrangian expression (28) recalculated with respect to the separately chosen charged particle ξ endowed with the uniform shell model geometric structure and generating this electromagnetic field.

Following the slightly modified well-known approach from [39] and reasonings from [7, 45] this Landau-Lifschitz type Lagrangian (28) can be recast (further in the

Gauss units) as

$$
\begin{aligned}
&\mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3r\big(\tfrac{|E|^2-|B|^2}{2} + \int_{\mathbb{R}^3} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) - \langle k(t), \tfrac{dr}{dt}\rangle = \\
&= \int_{\mathbb{R}^3} d^3r\big(\tfrac{1}{2}\langle -\nabla\varphi - \tfrac{1}{c}\tfrac{\partial A}{\partial t}, -\nabla\varphi - \tfrac{1}{c}\tfrac{\partial A}{\partial t}\rangle - \tfrac{1}{2}\langle \nabla\times(\nabla\times A), A\rangle\big)+ \\
&+ \int_{\mathbb{R}^3} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) - \langle k(t), dr/dt\rangle = \\
&= \int_{\mathbb{R}^3} d^3r\big(\tfrac{\langle -\nabla\varphi, E\rangle}{2} - \tfrac{1}{2c}\langle \tfrac{\partial A}{\partial t}, E\rangle - \tfrac{\langle A, \nabla\times B\rangle}{2}\big) + \int_{\mathbb{R}^3} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) - \langle k(t), \tfrac{dr}{dt}\rangle = \\
&= \int_{\mathbb{R}^3} d^3r\big(\tfrac{1}{2}\varphi\langle\nabla, E\rangle + \tfrac{1}{2c}\langle A, \tfrac{\partial E}{\partial t}\rangle - \tfrac{1}{2c}\langle A, J + \tfrac{\partial E}{\partial t}\rangle\big) + \int_{\mathbb{R}^3} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big)- \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r}\langle \varphi E + A\times B, dS^2_r\rangle - \langle k(t), dr/dt\rangle = \\
&= -\frac{1}{2}\int_{\Omega_+(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) - \langle k(t), \tfrac{dr}{dt}\rangle- \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r}\langle \varphi E + A\times B, dS^2_r\rangle = \\
&= -\frac{1}{2}\int_{\Omega_+(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) - \frac{1}{2}\int_{\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) + \frac{1}{2}\int_{\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big)+ \\
&+ \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) - \langle k(t), \tfrac{dr}{dt}\rangle - \frac{d}{dt}\int_{\mathbb{R}^3} d^3r\,\frac{\langle A, E\rangle}{2c} - \\
&- \lim_{r\to\infty}\int_{\mathbb{S}^2_r}\frac{\langle \varphi E + A\times B, dS^2_r\rangle}{2} = \frac{1}{2}\int_{\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big)- \\
&- \frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) - \langle k(t), \tfrac{dr}{dt}\rangle - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r}\langle \varphi E + A\times B, dS^2_r\rangle = \\
&= \frac{1}{2}\int_{\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big) + \frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\big(\tfrac{\langle J,A\rangle}{c} - \rho\varphi\big)- \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3r\langle A, E\rangle - \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}^2_r}\langle \varphi E + A\times B, dS^2_r\rangle,
\end{aligned}
$$

where we have introduced a (still not determined) radiation damping force $k(t) \in \mathbb{E}^3$, have denoted by $\Omega_+(\xi) := \operatorname{supp}\xi_+ \subset \mathbb{R}^3$ and $\Omega_-(\xi) := \operatorname{supp}\xi_- \subset \mathbb{R}^3$ the corresponding charge ξ supports, located on the electromagnetic field shadowed rear and electromagnetic field exerted front semispheres (see Fig.1) of the electron shell, respectively to its motion with the fixed velocity $u(t) \in \mathbb{E}^3$, as well as we denoted by $\mathbb{S}^2_r$ a two-dimensional sphere of radius $r \to \infty$.

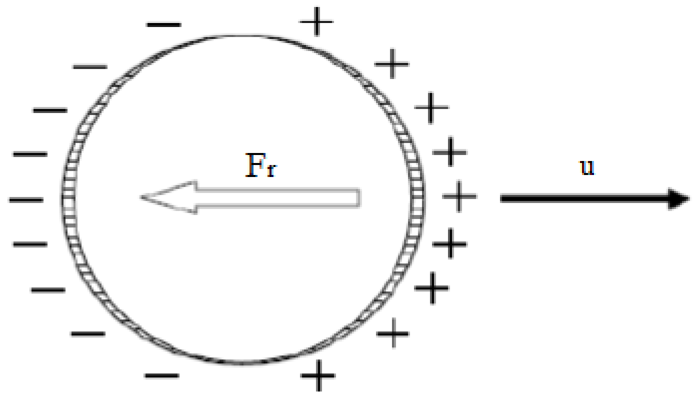

Fig.1 (The courtesy picture from [45])

Having naturally assumed that the radiated charged particle energy at infinity is negligible, the Lagrangian function $\mathcal{L}_{f-p}^{(t)}$becomes equivalent to

$$\mathcal{L}_{f-p}^{(t)} = \frac{1}{2}\int_{\Omega_-(\xi)} d^3r\big(\frac{\langle J,A\rangle}{c} - \rho\varphi\big) + \frac{1}{2c}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r(\langle J,A\rangle - \rho\varphi) - \langle k(t), \frac{dr}{dt}\rangle, \tag{29}$$

which we now need to additionally recalculate taking into account that the electromagnetic potentials $(\varphi, A) \in T^*(M^4)$ are retarded, generated by only the front part of the electron shell and given as $1/c^2 \to 0$ in the following expanded into Lienard-Wiechert series form:

$$\begin{aligned}
\varphi = & \int_{\mathbb{R}^3} d^3r' \frac{\rho(t',r')}{|r-r'|}\Big|_{t'=t-|r-r'|/c} = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3} d^3r' \frac{\rho(t-\varepsilon,r')}{|r-r'|} + \\
& + \lim_{\varepsilon\downarrow 0}\frac{1}{2c^2}\int_{\mathbb{R}^3} d^3r'|r-r'|\cdot\frac{\partial^2\rho(t-\varepsilon,r')}{\partial t^2} + \\
& + \lim_{\varepsilon\downarrow 0}\frac{1}{6c^3}\int_{\mathbb{R}^3} d^3r'|r-r'|^2\cdot\frac{\partial\rho(t-\varepsilon,r')}{\partial t} + O(\frac{1}{c^4}) = \\
& = \int_{\Omega_+(\xi)} d^3r'\frac{\rho(t,r')}{|r-r'|} + \frac{1}{2c^2}\int_{\Omega_+(\xi)} d^3r'|r-r'|\frac{\partial^2\rho(t,r')}{\partial t^2} + \\
& + \frac{1}{6c^3}\int_{\Omega_+(\xi)} d^3r'|r-r'|^2\frac{\partial\rho(t,r')}{\partial t} + O(\frac{1}{c^4}),
\end{aligned} \tag{30}$$

$$
\begin{aligned}
A = {} & \frac{1}{c}\int_{\mathbb{R}^3} d^3r' \frac{J(t',r')}{|r-r'|}\Big|_{t'=t-|r-r'|/c} = \lim_{\varepsilon\downarrow 0}\frac{1}{c}\int_{\mathbb{R}^3} d^3r' \frac{J(t-\varepsilon,r')}{|r-r'|} - \\
& - \lim_{\varepsilon\downarrow 0}\frac{1}{c^2}\int_{\mathbb{R}^3} d^3r' \frac{\partial J(t-\varepsilon,r')}{\partial t} + \lim_{\varepsilon\downarrow 0}\frac{1}{2c^3}\int_{\mathbb{R}^3} d^3r'|r-r'|\frac{\partial^2 J(t-\varepsilon,r')}{\partial t^2} + O(\tfrac{1}{c^4}) = \\
& = \frac{1}{c}\int_{\Omega_+(\xi)} d^3r' \frac{J(t,r')}{|r-r'|} - \frac{1}{c^2}\int_{\Omega_+(\xi)} d^3r' \frac{\partial J(t,r')}{\partial t} + \\
& + \frac{1}{2c^3}\int_{\Omega_+(\xi)} d^3r'|r-r'|\frac{\partial^2 J(t,r')}{\partial t^2} + O(\tfrac{1}{c^4}),
\end{aligned}
$$

where the current density $J(t,r) = \rho(t,r)dr/dt$ for all $t \in \mathbb{R}$ and $r \in \Omega(\xi) := \Omega_+(\xi)\cup\Omega_+(\xi) \simeq \mathbb{S}^2 := \operatorname{supp}\rho(t;r) \subset \mathbb{R}^3$, being the spherical compact support of the charged particle density distribution, and the limit $\lim_{\varepsilon\downarrow 0}$ *was treated physically, that is taking into account the assumed shell modell of the charged particle* ξ *and its corresponding charge density self interaction*. Moreover, the potentials (30) are both considered to be retarded and non singular, moving in space with the velocity $u \in T(\mathbb{R}^3)$ subject to the laboratory reference frame $\mathcal{K}_t$. As a result of simple enough calculations like in [36], making use of the expressions (30) one obtains that the Lagrangian function (29) brings about

$$
\mathcal{L}^{(t)}_{f-p} = \frac{\mathcal{E}_{es}}{2c^2}|u|^2 - \langle k(t), \tfrac{dr}{dt}\rangle, \tag{31}
$$

where we took into account that owing to the reasonings from [7, 45] the only front half the electric charge interacts with the whole virtually identical charge ξ, as well as made use of the following up to $O(1/c^4)$ limiting integral expressions:

$$
\begin{aligned}
& \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r'\rho(t,r')\rho(t,r') := \xi^2, \\
& \frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r' \frac{\rho(t,r')\rho(t,r')}{|r-r'|} := \mathcal{E}_{es}, \\
& \int_{\Omega_+(\xi)} d^3r\rho(t,r) \int_{\Omega_+(\xi)} d^3r' \frac{\rho(t;r')}{|r'-r|} = \tfrac{1}{2}\mathcal{E}_{es}, \\
& \int_{\Omega_-(\xi)} d^3r\rho(t,r) \int_{\Omega_-(\xi)} d^3r' \frac{\rho(t;r')}{|r'-r|} = \tfrac{1}{2}\mathcal{E}_{es}, \\
& \int_{\Omega_-(\xi)} d^3r\rho(t,r) \int_{\Omega_+(\xi)} d^3r' \frac{\rho(t;r')}{|r-r'|}\left|\frac{\langle r'-r,u\rangle}{|r'-r|}\right|^2 = \tfrac{1}{6}\mathcal{E}_{es}|u|^2, \\
& \int_{\Omega_+(\xi)} d^3r\rho(t,r) \int_{\Omega_+(\xi)} d^3r' \frac{\rho(t;r')}{|r-r'|}\left|\frac{\langle r'-r,u\rangle}{|r'-r|}\right|^2 = \tfrac{1}{6}\mathcal{E}_{es}|u|^2.
\end{aligned} \tag{32}
$$

To obtain the corresponding evolution equation for our charged particle ξ we need, within the Feynman proper time paradigm, to transform the Lagrangian function (31) to the one with respect to the proper time reference frame $\mathcal{K}_\tau$:

$$
\mathcal{L}^{(\tau)}_{f-p} = \tfrac{1}{2}m_{es}|\dot{r}|^2/\sqrt{1+|\dot{r}|^2/c^2} - \langle k(t), \dot{r}\rangle,
$$

where, for brevity, we have denoted by $\dot{r} := dr/d\tau$ the charged particle velocity with respect to the proper reference frame $\mathcal{K}_\tau$ and by, definition, $m_{es} := \mathcal{E}_{es}/c^2$ its so called electrostatic mass with respect to the laboratory reference frame $\mathcal{K}_t$.

Thus, the generalized charged particle ξ momentum (up to $O(1/c^4)$) equals

$$\begin{aligned} \pi_p := \partial \mathcal{L}^{(\tau)}_{f-p}/\partial \dot{r} = \frac{m_{es}\dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} - \frac{m_{es}|\dot{r}|^2\dot{r}}{2c^2(1+|\dot{r}|^2/c^2)^{3/2}} - k(t) = \\ = m_{es}u(1-\tfrac{|u|^2}{2c^2}) - k(t) \simeq m_{es}u\sqrt{1-\tfrac{|u|^2}{c^2}} - k(t) = \bar{m}_{es}u - k(t), \end{aligned} \tag{33}$$

where we denoted, as before, by $u := dr/dt$ the charged particle ξ velocity with respect to the laboratory reference frame $\mathcal{K}_t$ and put, by definition,

$$\bar{m}_{es} := m_{es}\sqrt{1-|u|^2} \tag{34}$$

its mass parameter $\bar{m}_{es} \in \mathbb{R}_+$ with respect to the proper reference frame $\mathcal{K}_\tau$.

The generalized momentum (34) satisfies with respect to the proper reference frame $\mathcal{K}_\tau$ the evolution equation $d\pi_p/d\tau := \partial \mathcal{L}^{(\tau)}_{f-p}/\partial r = 0$, being equivalent, with respect to the laboratory reference frame $\mathcal{K}_t$, to the Lorentz type equation

$$\frac{d}{dt}(\bar{m}_{es}u) = \frac{dk(t)}{dt}. \tag{35}$$

The evolution equation (35) allows the corresponding canonical Hamiltonian formulation on the phase space $T^*(\mathbb{R}^3)$ with the Hamiltonian function

$$\begin{aligned} H_{f-p} : = \langle \pi_p, r\rangle - \mathcal{L}^{(\tau)}_{f-p} = \langle \tfrac{m_{es}\dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} - \tfrac{m_{es}|\dot{r}|^2\dot{r}}{2c^2(1+|\dot{r}|^2/c^2)^{3/2}} - k(t), \dot{r}\rangle - \\ - \tfrac{1}{2}m_{es}|\dot{r}|^2/\sqrt{1+|\dot{r}|^2/c^2} + \langle k(t), \dot{r}\rangle = \tfrac{1}{2}\bar{m}_{es}|u|^2, \end{aligned} \tag{36}$$

naturally looking and satisfying up to $O(1/c^4)$ for all τ and $t \in \mathbb{R}$ the conservation conditions $\frac{d}{d\tau}H_{f-p} = 0 = \frac{d}{dt}H_{f-p}$. Looking at the equation (35) and (36), one can state that the physically observable inertial charged particle ξ mass parameter $m_{phys} := \bar{m}_{es}$, being exactly equal to the relativistic charged particle ξ electromagnetic mass, as it was assumed by H. Lorentz and Abraham.

To determine the damping radiation force $k(t) \in \mathbb{E}^3$, we can make use of the Lorentz type force expression (29) and obtain, similarly to [36], up to $O(1/c^4)$ accuracy, the resulting self-interacting Abraham-Lorentz type force expression. Thus, owing to the zero net force condition, we have that

$$\frac{d\pi_p}{dt} + F_s = 0, \tag{37}$$

where the Lorentz force

$$
\begin{aligned}
F_s = &-\tfrac{1}{2c}\int_{\Omega_-(\xi)} d^3r\rho(t,r)\tfrac{d}{dt}A(t,r) - \tfrac{1}{2c}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\rho(t,r)\tfrac{d}{dt}A(t,r)- \\
&-\tfrac{1}{2}\int_{\Omega_-(\xi)} d^3r\rho(t,r)\nabla\varphi(t,r)\big(1-\left|\tfrac{u}{c}\right|^2\big)- \\
&-\tfrac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3r\rho(t,r)\nabla\varphi(t,r)\big(1-\left|\tfrac{u}{c}\right|^2\big).
\end{aligned}
\tag{38}
$$

This expression easily follows from the least action condition $\delta S^{(t)} = 0$, where $S^{(t)} := \int_{t_1}^{t_2} \mathcal{L}^{(t)}_{f-p} dt$ with the Lagrangian function given by the derived above Landau-Lifschitz type expression (32), and the potentials $(\varphi, A) \in T^*(M^4)$ given by the Lienard-Wiechert expressions (30). Followed by calculations similar to those of [36, 6], from (38) and (30) one can obtain, within the assumed before uniform shell electron model, for small $|u/c| \ll 1$ and slow enough acceleration that

$$
\begin{aligned}
F_s = &\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^n}\big(1-\left|\tfrac{u}{c}\right|^2\big)\int_{\Omega_+(\xi)} d^3r'\tfrac{\partial^n}{\partial t^n}\rho(t,r')\nabla|r-r'|^{n-1}\times \\
&\times\Big[\int_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\Big]+ \\
&+\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^{n+2}}\int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r')\times \\
&\times\Big[\int_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\Big] = \\
= &\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^{n+2}}(1-\left|\tfrac{u}{c}\right|^2)\int_{\Omega_+(\xi)} d^3r'\frac{\partial^{n+2}}{\partial t^{n+2}}\rho(t,r')\nabla|r-r'|^{n+1}\times \\
&\times\Big[\int_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\Big]+ \\
&+\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^{n+2}}\int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,r')\times \\
&\times\Big[\int_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\Big].
\end{aligned}
$$

The relationship above can be rewritten, owing to the charge continuity equation (20), giving rise to the radiation force expression

$$
\begin{aligned}
F_s = &\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{2n!c^{n+2}}(1-|\tfrac{u}{c}|^2)\Big[\int_{\Omega_-(\xi)} \rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \rho(t,r)d^3r(\cdot)\Big]\times \\
&\times \int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\Big(\tfrac{J(t,r')}{n+2} + \tfrac{n-1}{n+2}\tfrac{\langle r-r', J(t,r')\rangle(r-r')}{|r-r'|^2}\Big)+ \\
&+ \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^{n+2}}\Big[\int_{\Omega_-(\xi)} \rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \rho(t,r)d^3r(\cdot)\Big]\times \\
&\times \int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r') = \\
&= \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^{n+2}}(1-|\tfrac{u}{c}|^2)\Big[\int_{\Omega_-(\xi)} \rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \rho(t,r)d^3r(\cdot)\Big]\times \\
&\times \int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\Big(\tfrac{J(t,r')}{n+2} + \tfrac{n-1}{n+2}\tfrac{|r-r',u|^2J(t,r')}{|r-r'|^2|u|^2}\Big)+ \\
&+ \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^{n+2}}\Big[\int_{\Omega_-(\xi)} \rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \rho(t,r)d^3r(\cdot)\Big]\times \\
&\times \int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r').
\end{aligned}
$$

Now, having applied to this equation the rotational symmetry property for calculation of the internal integrals, one easily obtains in the case of a charged particle ξ uniform shell model that

$$
\begin{aligned}
F_s = &\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{2n!c^{n+2}}(1-|\tfrac{u}{c}|^2)\Big[\int_{\Omega_-(\xi)} \rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \rho(t,r)d^3r(\cdot)\Big]\times \\
&\qquad\times \int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\Big(\tfrac{J(t,r')}{n+2} + \tfrac{(n-1)J(t,r')}{3(n+2)}\Big)+ \\
&+ \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{2n!c^{n}}\Big[\int_{\Omega_-(\xi)} \rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} \rho(t,r)d^3r(\cdot)\Big]\times \\
&\qquad\times \int_{\Omega_+(\xi)} d^3r'\tfrac{|r-r'|^{n+1}}{c^2}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r')
\end{aligned}
$$

$$
\begin{aligned}
F_s = \frac{d}{dt}\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{6n!c^{n+2}}\Big[\int_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\Big] \times \\
\times\int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^n}{\partial t^n}J(t,r') - \\
-\frac{d}{dt}\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n}\,|u|^2}{6n!c^{n+4}})\Big[\int_{\Omega_-(\xi)}\rho(t,r)d^3r(\cdot) + \int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,r)d^3r(\cdot)\Big]\times \\
\times\int_{\Omega_+(\xi)} d^3r'|r-r'|^{n-1}\tfrac{\partial^n}{\partial t^n}J(t,r').
\end{aligned}
$$

Now, having took into account the integral expressions (32), one finds from the above equation that up to the $O(1/c^4)$ accuracy the following radiation reaction force expression

$$
\begin{aligned}
F_s &= -\tfrac{d}{dt}\big(\tfrac{\mathcal{E}_{es}}{c^2}u\big) + \tfrac{d}{dt}\big(\tfrac{\mathcal{E}_{es}}{2c^2}|\tfrac{u}{c}|^2u(t)\big) + \tfrac{2\xi^2}{3c^3}\tfrac{d^2u}{dt^2} + O(\tfrac{1}{c^4}) = \\
&= -\tfrac{d}{dt}\big(m_{es}(1-\tfrac{1}{2}|\tfrac{u}{c}|^2)u\big) + \tfrac{2\xi^2}{3c^3}\tfrac{d^2u}{dt^2} + O(\tfrac{1}{c^4}) = \\
&= -\tfrac{d}{dt}\big(m_{es}\sqrt{1-|\tfrac{u}{c}|^2}u\big) + \tfrac{2\xi^2}{3c^3}\tfrac{d^2u}{dt^2} + O(\tfrac{1}{c^4}) = \\
&= -\tfrac{d}{dt}(\bar{m}_{es}u - \tfrac{2\xi^2}{3c^3}\tfrac{du}{dt}) + O(\tfrac{1}{c^4})
\end{aligned}
$$

holds. We mention here that following the reasonings from [7, 45, 56], in the expressions above there is taken into account an additional hidden and the velocity $u \in T(\mathbb{R}^3)$ directed electrostatic Coulomb surface self-force, acting only on the *front half part* of the spherical electron shell. As a result, from (37), (38) and the relationship (33) one obtains that the electron momentum

$$
\pi_p := \bar{m}_{es}u - \frac{2\xi^2}{3c^3}\frac{du}{dt} = \bar{m}_{es}u - k(t), \tag{39}
$$

thereby defining both the radiation reaction momentum $k(t) = \frac{2\xi^2}{3c^3}\frac{du}{dt}$ and the corresponding radiation reaction force $F_r = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(\frac{1}{c^4})$, coincides exactly with the classical Abraham–Lorentz–Dirac expression. Moreover, it also follows that the observable physical shell model electron inertial mass

$$
m_{ph} = m_{es} := \mathcal{E}_{es}/c^2, \tag{40}
$$

being completely of the electromagnetic origin, giving rise to the final force expression

$$
\frac{d}{dt}(m_{ph}u) = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(\tfrac{1}{c^4}). \tag{41}
$$

This means, in particular, that the real physically observed "inertial" mass m_{ph} of an electron within the uniform shell model is strongly determined by its electromagnetic self-interaction energy $\mathcal{E}_{es}$. A similar statement there was recently demonstrated using completely different approaches in [56, 45], based on the vacuum Casimir effect considerations. Moreover, the assumed above boundedness of the electrostatic self-energy $\mathcal{E}_{es}$ appears to be completely equivalent to the existence of so-called intrinsic Poincaré type "*tensions*", analyzed in [7, 45], and to the existence of a special compensating Coulomb "*pressure*", suggested in [56], guaranteeing the observable electron stability.

Remark 2.1. Some years ago there was suggested in the work [41] a "solution" to the mentioned before "4/3-electron mass" problem, expressed by the physical mass relationship (40) and formulated more than one hundred years ago by H. Lorentz and M. Abraham. To the regret, the above mentioned "solution" appeared to be fake that one can easily observe from the main not correct assumptions on which the work [41] has been based: the first one is about the particle-field momentum conservation, taken in the form

$$\frac{d}{dt}(p + \xi A) = 0, \tag{42}$$

and the second one is a speculation about the $1/2$-coefficient imbedded into the calculation of the Lorentz type self-interaction force

$$F := -\frac{1}{2c}\int_{\mathbb{R}^3} d^3r\rho(t;r)\partial A(t;r)/\partial t, \tag{43}$$

being not correctly argued by the reasoning that the expression (43) represents "... the interaction of a given element of charge with all other parts, otherwise we count twice that reciprocal action" (cited from [41], page 2710). This claim is fake as there was not taken into account the important fact that the interaction in the integral (43) is, in reality, retarded and its impact into it should be considered as that calculated for two virtually different charged particles, as it has been done in the classical works of H. Lorentz and M. Abraham. Subject to the first assumption (42) it is enough to recall that a vector of the field momentum $\xi A \in \mathbb{E}^3$ is not independent and is, within the charged particle model considered, strongly related with the local flow of the electromagnetic potential energy in the Lorentz constraint form:

$$\frac{\partial\varphi}{\partial t} + \langle\nabla, A\rangle = 0,$$

under which there hold the exploited in the work [41] the Lienard-Wiechert expressions (29) potentials for calculation of the integral (43). Thus, the equation (42), following the classical Newton second law, should be replaced by

$$\frac{d}{dt'}(p' + \xi A') = -\nabla(\xi\varphi'), \tag{44}$$

written with respect to the reference frame $\mathcal{K}(t';r)$ subject to which the charged particle ξ is at rest. Taking into account that with respect to the laboratory reference frame $\mathcal{K}_t$ there hold the relativistic relationships $dt = dt'\sqrt{1 - |u|^2/c^2}$ and

$\varphi' = \varphi\sqrt{1-|u|^2/c^2}$, from (44) one easily obtains that

$$\tfrac{d}{dt}(p+\xi A) = -\xi\nabla\varphi(1-|\tfrac{u}{c}|^2) = -\xi\nabla\varphi+\tfrac{\xi}{c}\nabla\langle u, \tfrac{u\varphi}{c}\rangle = -\xi\nabla\varphi+\tfrac{\xi}{c}\nabla\langle u, A\rangle. \quad (45)$$

Here we made use of the well-known relationship $A = u\varphi/c$ for the vector potential generated by this charged particle ξ moving in space with the velocity $u \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}_t$. Based now on the equation (45) one can derive the final expression for the evolution of the charged particle ξ momentum:

$$\begin{aligned}\frac{dp}{dt} &= -\xi\nabla\varphi - \tfrac{\xi}{c}\tfrac{dA}{dt} + \tfrac{\xi}{c}\nabla\langle u, A\rangle = -\xi\nabla\varphi - \tfrac{\xi}{c}\partial A/\partial t - \tfrac{\xi}{c}\langle u, \nabla\rangle A + \tfrac{\xi}{c}\nabla\langle u, A\rangle = \\ &= \xi E + \tfrac{\xi}{c}u\times(\nabla\times A) = \xi E + \tfrac{\xi}{c}u\times B,\end{aligned}$$

that is exactly the well known Lorentz force expression, used in the works of H. Lorentz and M. Abraham.

2.2. The 4/3-electron mass problem revisiting

Recently enough there appeared other interesting works devoted to this "4/3-electron mass" problem, amongst which we would like to mention [45, 56, 47, 48], whose argumentations are close to each other and based on the charged shell electron model, within which there is assumed a virtual interaction of the electron with the ambient "dark" radiation energy. The latter was first clearly demonstrated in [56], where a suitable compensation mechanism of the related singular electrostatic Coulomb electron energy and the wide band vacuum electromagnetic radiation energy fluctuations deficit inside the electron shell was shown to be harmonically realized as the electron shell radius $a \to 0$. Moreover, this compensation happens exactly when the induced outward directed electrostatic Coulomb pressure on the whole electron coincides, up to the sign, with that induced by the mentioned above vacuum electromagnetic energy fluctuations outside the electron shell, since there was manifested their absence inside the electron shell.

Really, the outward directed electrostatic spatial Coulomb pressure on the electron equals

$$\eta_{coul} := \lim_{a\to 0}\frac{\varepsilon_0|E|^2}{2}\Big|_{r=a} = \lim_{a\to 0}\frac{\xi^2}{32\varepsilon_0\pi^2a^4}, \tag{46}$$

where $E = \frac{\xi r}{4\pi\varepsilon_0|r|^3} \in \mathbb{E}^3$ is the electrostatic field at point $r \in \mathbb{R}$ subject to the electron center at the point $r = 0 \in \mathbb{R}$. The related inward directed vacuum electromagnetic fluctuations spatial pressure equals

$$\eta_{vac} := \lim_{\Omega\to\infty}\frac{1}{3}\int_0^{\Omega} d\mathcal{E}(\omega), \tag{47}$$

where $d\mathcal{E}(\omega)$ is the electromagnetic energy fluctuations density for a frequency $\omega \in \mathbb{R}_+$, and $\Omega \in \mathbb{R}_+$ is the corresponding electromagnetic frequency cutoff. The integral

(47) can be calculated if to take into account the quantum statistical recipe [26, 33, 10] that

$$d\mathcal{E}(\omega) := \hbar\omega \frac{d^3 p(\omega)}{h^3}, \tag{48}$$

where the Plank constant $h := 2\pi\hbar$ and the electromagnetic wave momentum $p(\omega) \in \mathbb{E}^3$ satisfies the relativistic relationship

$$|p(\omega)| = \hbar\omega/c. \tag{49}$$

Whence by substituting (49) into (48) one obtains $d\mathcal{E}(\omega) = \frac{\hbar\omega^3}{2\pi^2c^3}d\omega$, which entails, owing to (47), the following vacuum electromagnetic energy fluctuations spatial pressure

$$\eta_{vac} = \lim_{\Omega\to\infty} \frac{\hbar\Omega^4}{24\pi^2c^3}. \tag{50}$$

For the charged electron shell model to be stable at rest it is necessary to equate the inward (50) and outward (46) spatial pressures:

$$\lim_{\Omega\to\infty} \frac{\hbar\Omega^4}{24\pi^2c^3} = \lim_{a\to 0} \frac{\xi^2}{32\varepsilon_0\pi^2a^4},$$

giving rise to the balance electron shell radius $a_b \to 0$ limiting condition:

$$a_b = \lim_{\Omega\to\infty} \Big[\Omega^{-1}\Big(\frac{3\xi^2c^2}{2\hbar}\Big)^{1/4}\Big].$$

Simultaneously we can calculate the corresponding Coulomb and electromagnetic fluctuations energy deficit inside the electron shell:

$$\Delta W_b := \frac{1}{2}\int_{a_b}^{\infty} \varepsilon_0|E|^2 d^3r - \int_0^{a_b} d^3r \int_0^{\Omega} d\mathcal{E}(\omega) = \frac{\xi^2}{8\pi\varepsilon_0 a_b} - \frac{\hbar\Omega^4 a_b^3}{6\pi c^3} = 0,$$

additionally ensuring the electron shell model stability.

Another important consequence from this pressure-energy compensation mechanism can be derived concerning the electron inertial mass $m_{ph} \in \mathbb{R}_+$, entering the momentum expression (39) in the case of the electron slow enough movement. Namely, following the reasonings from [45], one can observe that during the electron movement there arises an additional hidden not compensated and velocity $u \in T(\mathbb{R}^3)$ directed electrostatic Coulomb surface self-pressure acting only on the *front half part* of the electron shell and equal to

$$\eta_{surf} := \frac{|E\xi|}{4\pi a_b^2}\frac{1}{2} = \frac{\xi^2}{32\pi\varepsilon_0 a_b^4},$$

coinciding, evidently, with the already compensated outward directed electrostatic Coulomb spatial pressure (46). As, evidently, during the electron motion in space its surface electric current energy flow is not vanishing [45], one ensues that the electron momentum gains an additional mechanical impact, which can be expressed as

$$\pi_\xi := -\eta_{surf}\frac{4\pi a_b^3}{3c^2}u = -\frac{1}{3}\frac{\xi^2}{8\pi\varepsilon_0 a_b c^2}u = -\frac{1}{3}\bar{m}_{es}u, \tag{51}$$

where we took into account that within this electron shell model the corresponding electrostatic electron mass equals its electrostatic energy $\bar{m}_{es} = \frac{\xi^2}{8\pi\varepsilon_0 a_b c^2}$.

Thus, one can claim that, owing to the structural stability of the electron shell model, its generalized self-interaction momentum $\pi_p \in T^*(\mathbb{R}^3)$ gains during the movement with velocity $u = dr/dt \in T(\mathbb{R}^3)$ the additional backward directed hidden impact (51), which can be identified with the back-directed momentum component

$$\pi_\xi = -\frac{1}{3}\bar{m}_{es}u, \tag{52}$$

complementing the classical [36, 6] momentum expression $\pi_p = \frac{4}{3}\bar{m}_{es}u$, which can be easily obtained from the Lagrangian expression, if one not to take into account the shading property of the moving uniform shell electron model. Then, owing to the additional momentum (52), the full momentum becomes as $\pi_p = \pi_\xi + \frac{4}{3}\bar{m}_{es}u = (-\frac{1}{3}\bar{m}_{es} + \frac{4}{3}\bar{m}_{es})u = \bar{m}_{es}u$, coinciding with that of (33) modulo the radiation reaction momentum $k(t) = \frac{2\xi^2}{3c^3}\frac{du}{dt}$, strongly supporting the electromagnetic energy origin of the electron inertial mass for the first time conceived by H. Lorentz and M. Abraham.

3. Comments

The electromagnetic mass origin problem was reanalyzed in details within the Feynman proper time paradigm and related vacuum field theory approach by means of the fundamental least action principle and the Lagrangian and Hamiltonian formalisms. The resulting electron inertia appeared to coincide in part, in the quasi-relativistic limit, with the momentum expression obtained more than one hundred years ago by M. Abraham and H. Lorentz [1, 40], yet it proved to contain an additional hidden impact owing to the imposed electron stability constraint, which was taken into account in the original action functional as some preliminarily undetermined constant component. As it was demonstrated in [56, 45], this stability constraint can be successfully realized within the charged shell model of electron at rest, if to take into account the existing ambient electromagnetic "dark" energy fluctuations, whose inward directed spatial pressure on the electron shell is compensated by the related outward directed electrostatic Coulomb spatial pressure as the electron shell radius satisfies some limiting compatibility condition. The latter also allows to compensate simultaneously the corresponding electromagnetic energy fluctuations deficit inside the electron shell, thereby forbidding the external energy to flow into the electron.

In contrary to the lack of energy flow inside the electron shell, during the electron movement the corresponding internal momentum flow is not vanishing owing to the nonvanishing hidden electron momentum flow caused by the surface pressure flow and compensated by the suitably generated surface electric current flow. As it was shown, this backward directed hidden momentum flow makes it possible to justify the corresponding self-interaction electron mass expression and to state, within the electron shell model, the fully electromagnetic electron mass origin, as it has been conceived by H. Lorentz and M. Abraham and strongly supported by R. Feynman in his Lectures [26]. This consequence is also independently supported by means of the least action approach, based on the Feynman proper time paradigm and the suitably calculated regularized retarded electric potential impact into the charged particle Lagrangian function.

The charged particle radiation problem, revisited in this Section, allowed to conceive the explanation of the charged particle mass as that of a compact and stable object which should be exerted by a vacuum field self-interaction energy. The latter can be satisfied iff the expressions (32) hold, thereby imposing on the intrinsic charged particle structure [42] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (40), the electrostatic potential energy being of the self-interaction origin, contributes into the inertial mass as its main relativistic mass component.

There exist different relativistic generalizations of the force expression (41), which suffer the common physical inconsistency related to the no radiation effect of a charged particle in uniform motion.

Another deeply related problem to the radiation reaction force analyzed above is the search for an explanation to the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, strongly based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

4. Supplement: The classical relativistic electrodynamics backgrounds: a charged point particle analysis

It is well known [39, 26, 49, 4] that the classical relativistic least action principle for a point charged particle ξ in the Minkowski space-time $M^4 \simeq \mathbb{E}^3 \times \mathbb{R}$ is formulated on a time interval $[t_1, t_2] \subset \mathbb{R}$ (in the light speed units) as $\delta S^{(t)} = 0$ where

$$S^{(t)} := \int_{\tau(t_1)}^{\tau(t_2)} (-m_0 d\tau - \xi \langle \mathcal{A}, dx\rangle_{M^4}) = \int_{s(t_1)}^{s(t_2)} (-m_0 \langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} - \xi \langle \mathcal{A}, \dot{x}\rangle_{M^4}) ds. \quad (53)$$

Here $\delta x(s(t_1)) = 0 = \delta x(s(t_2))$ are the boundary constraints, $m_0 \in \mathbb{R}_+$ is the so called particle *rest mass*, the 4-vector $x := (t, r) \in M^4$ is the particle location in M^4, $\dot{x} := dx/ds \in T(M^4)$ is the particle Euclidean "four-vector" velocity with respect to a laboratory reference frame $\mathcal{K}$, parameterized by means of the Minkowski space-time parameters $(s(t), r) \in M^4$ and related to each other by means of the infinitesimal Lorentz interval relationship

$$d\tau := \langle dx, dx \rangle_{M^4}^{1/2} := ds \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2}, \tag{54}$$

$\mathcal{A} \in T^*(M^4)$ is an external electromagnetic 4-vector potential, satisfying the classical Maxwell equations [49, 39, 26, 27], the sign $\langle \cdot, \cdot \rangle_{M^4}$ means the corresponding scalar product in a finite-dimensional vector space $T(M^4) \simeq T^*(M^4)$, notations $T(M^4)$ and $T^*(M^4)$ are, respectively, the tangent and cotangent spaces [2, 3, 60, 21, 32] to the Minkowski space M^4. In particular, $\langle \dot{x}, \dot{x} \rangle >_{M^4} := (dt/ds)^2 - \langle dr/ds, dr/ds \rangle_{\mathbb{E}^3}$ for any $x := (t, r) \in M^4$.

The subintegral expression in (53)

$$\mathcal{L}^{(t)} := -m_0 \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} - \xi \langle \mathcal{A}, \dot{x} \rangle_{M^4} \tag{55}$$

is the related Lagrangian function, whose first part is proportional to the particle world line length *with respect to the proper reference frame* $\mathcal{K}_r$ and the second part is proportional to the pure electromagnetic particle-field interaction *with respect to the Minkowski laboratory reference frame* $\mathcal{K}$. Moreover, the positive rest mass parameter $m_0 \in \mathbb{R}_+$ is introduced into (55) as *an external physical ingredient*, also describing the point particle with respect to the proper reference frame $\mathcal{K}_r$, *yet its physical essence remains to be hidden*. The electromagnetic 4-vector potential $\mathcal{A} \in T^*(M^4)$ is at the same time expressed as a 4-vector, constructed and measured with respect to the Minkowski laboratory reference frame $\mathcal{K}$ that looks *from physical point of view* enough controversial, since the action functional (53) is forced to be extremal *with respect to the laboratory reference frame* $\mathcal{K}$. This, in particular, means that the real physical motion of our charged point particle, being realized with respect to the proper reference frame $\mathcal{K}_r$, somehow depends on an external observation data [26, 27, 25, 40, 17] with respect to the occasionally chosen laboratory reference frame $\mathcal{K}$. This aspect was never discussed in the physical literature except of very interesting reasonings by R. Feynman in [26], who argued that the relativistic expression for the classical Lorentz force has a physical sense only with respect to the Euclidean rest reference frame $\mathcal{K}_r$ variables $(\tau, r) \in \mathbb{E}^4$ related with the Minkowski laboratory reference frame $\mathcal{K}$ parameters $(t, r) \in M^4$ by means of the infinitesimal relationship

$$d\tau := \langle dx, dx \rangle_{M^4}^{1/2} = dt \sqrt{1 - |u|^2}, \tag{56}$$

where $u := dr/dt \in T(\mathbb{E}^3)$ is the point particle velocity with respect to the reference frame $\mathcal{K}$.

It is worth to point out here that to be correct, it would be necessary to include still into the action functional the additional part describing the electromagnetic field itself. But, in general, this part is not taken into account, since there is generally assumed [15, 38, 37, 10, 61, 42, 43, 46] that the charged particle influence on the electromagnetic field is negligible. This is true, if the particle charge value ξ is very small and the support $\operatorname{supp} \mathcal{A} \subset M^4$ of the electromagnetic 4-vector potential is compact. It is clear that in the case of two interacting to each other charged particles the above assumption can not be applied, as it is necessary to take into account the relative motion of two particles and the respectively changing delay interaction energy. This aspect of the action functional choice problem appears to be very important when one tries to analyze the related Lorentz type forces exerted by charged particles on each other. We will return to this problem in a separate section below.

Having calculated the least action condition (53), we easily obtain from (55) the classical relativistic dynamical equations

$$\begin{aligned} dP/ds &:= -\partial \mathcal{L}^{(t)}/\partial x = -\xi \nabla \langle \mathcal{A}, \dot{x} \rangle_{M^4}, \\ P &:= -\partial \mathcal{L}^{(t)}/\partial \dot{x} = m_0 \, \dot{x} \, \langle \dot{x}, \dot{x} \rangle_{M^4}^{-1/2} + \xi \mathcal{A}, \end{aligned} \tag{57}$$

where by $P \in T^*(M^4)$ we denoted the common particle-field generalized momentum of the interacting system.

Now at $s = t \in \mathbb{R}$ by means of the standard infinitesimal change of variables (56) we can easily obtain from (57), respectively, the classical Lorentz force expression

$$\frac{dp}{dt} = \xi E + \xi u \times B \tag{58}$$

with the relativistic particle momentum and mass

$$p := mu, \quad m := m_0 / \sqrt{1 - |u|^2}, \tag{59}$$

the electric field $E := -\partial A/\partial t - \nabla \varphi$ and the magnetic field $B := \nabla \times A$, where we have expressed the electromagnetic 4-vector potential as $\mathcal{A} := (\varphi, A) \in T^*(M^4)$.

The Lorentz force (58), owing to our preceding assumption, means the force exerted by the external electromagnetic field on our charged point particle, whose charge ξ is so negligible that it does not exert the influence on the field. This fact becomes very important if we try to make use of the Lorentz force expression (58) for the case of two interacting to each other charged particles, since then one can not assume that our charge ξ exerts negligible influence on other charged particle. Thus, the corresponding Lorentz force between two charged particles should be suitably altered. Nonetheless, the modern physics [14, 20, 39, 16, 18, 34, 5, 19, 35] did not make this naturally needed Lorentz force modification and there is everywhere used the classical expression (58). This situation was observed and analyzed concerning the related physical aspects in [57], having shown that the electromagnetic Lorentz force between two moving charged particles can be modified in such a way that it ceases to be dependent on their relative motion contrary to the classical relativistic case.

To the regret, the least action principle approach to analyzing the Lorentz force structure was in [57] completely ignored that gave rise to some incorrect and physically not motivated statements concerning mathematical physics backgrounds of the modern electrodynamics. To make the problem more transparent we will analyze it in the section below from the vacuum field theory approach recently devised in [53, 54, 12].

4.1. Supplement: the classical relativistic invariant least action principle physical backgrounds

Consider the least action principle (53) and observe that the extremality condition $\delta S^{(t)} = 0$, $\delta x(s(t_1)) = 0 = \delta x(s(t_2))$, is calculated with respect to the laboratory reference frame $\mathcal{K}$, whose point particle coordinates $(r, t) \in M^4$ are parameterized by means of an arbitrary parameter $s \in \mathbb{R}$ owing to expression (54). Recalling now the definition of the invariant proper reference frame $\mathcal{K}_r$ time parameter (56), we obtain that at the critical parameter value $s = \tau \in \mathbb{R}$ the action functional (53) on the fixed interval $[\tau_1, \tau_2] \subset \mathbb{R}$ turns into

$$S^{(t)} = \int_{\tau_1}^{\tau_2} (-m_0 - \xi \langle \mathcal{A}, \dot{x} \rangle_{M^4}) d\tau \tag{60}$$

under the additional constraint

$$\langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} = 1, \tag{61}$$

where, by definition, $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$.

The expressions (60) and (61) need some comments since the corresponding to (60) Lagrangian function

$$\mathcal{L}^{(t)} := -m_0 - \xi \langle \mathcal{A}, \dot{x} \rangle_{M^4} \tag{62}$$

depends only virtually on the unobservable rest mass parameter $m_0 \in \mathbb{R}_+$ and, evidently, it has no direct impact into the resulting particle dynamical equations following from the condition $\delta S^{(t)} = 0$. Nonetheless, the rest mass *springs up* as a suitable Lagrangian multiplier owing to the imposed constraint (61). To demonstrate this consider the extended Lagrangian function (62) in the form $\mathcal{L}_\lambda^{(t)} := -m_0 - \xi \langle \mathcal{A}, \dot{x} \rangle_{M^4} - \lambda(\langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} - 1)$, where $\lambda \in \mathbb{R}$ is a suitable Lagrangian multiplier. The resulting Euler equations look as

$$P_r := \partial \mathcal{L}_\lambda^{(t)}/\partial \dot{r} = \xi A + \lambda \dot{r}, \qquad P_t := \partial \mathcal{L}_\lambda^{(t)}/\partial \dot{t} = -\xi \varphi - \lambda \dot{t},$$
$$\partial \mathcal{L}_\lambda^{(t)}/\partial \lambda = \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} - 1 = 0, \qquad dP_r/d\tau = \xi \nabla_r \langle A, \dot{r} \rangle_{\mathbb{E}^3} - \xi \dot{t} \nabla_r \varphi,$$
$$dP_t/d\tau = \xi \langle \partial A/\partial t, \dot{r} \rangle_{\mathbb{E}^3} - \xi \dot{t} \partial \varphi/\partial t,$$

giving rise, owing to relationship (56), to the following dynamical equations:

$$\tfrac{d}{dt}(\lambda u\dot{t}) = \xi E + \xi u \times B, \quad \tfrac{d}{dt}(\lambda \dot{t}) = \xi \langle E, u\rangle_{\mathbb{E}^3}, \tag{63}$$

where we denoted by $E := -\partial A/\partial t - \nabla\varphi,\ \ B = \nabla \times A$ the corresponding electric and magnetic fields. As a simple consequence of (63) one obtains

$$\frac{d}{dt}\ln(\lambda\dot{t}) + \frac{d}{dt}\ln\sqrt{1-|u|^2} = 0,$$

being equivalent for all $t \in \mathbb{R}$, owing to relationship (56), to the relationship $\lambda\dot{t}\sqrt{1-|u|^2} = \lambda := m_0$, where $m_0 \in \mathbb{R}_+$ is a constant, which could be interpreted as the rest mass of our charged point particle ξ. Really, the first equation of (63) can be rewritten as $dp/dt = \xi E + \xi u \times B$, where we denoted $p := mu$, $m := \lambda\dot{t} = m_0/\sqrt{1-|u|^2}$, coinciding exactly with that of (56).

Thereby, we retrieved here all of the results obtained in the section above, making use of the action functional (60), represented with respect to the proper reference frame $\mathcal{K}_r$ under constraint (61). During these derivations, we faced with a very delicate *inconsistency property* of definition of the action functional $S^{(t)}$, defined *with respect to the proper reference frame* $\mathcal{K}_r$, but depending on the external electromagnetic potential function $\mathcal{A} : M^4 \to T^*(M^4)$, constructed exceptionally *with respect to the laboratory reference frame* $\mathcal{K}$. Namely, this potential function, as a physical observable quantity, is defined and, respectively, measurable only *with respect to the fixed laboratory reference frame* $\mathcal{K}$.

Thus, the corresponding Lorentz invariant action functional, in reality, should be from the very beginning written physically correct as

$$S^{(\tau)} = \int_{t(\tau_1)}^{t(\tau_2)} (-\xi\langle\mathcal{A}, \dot{x}\rangle_{\mathbb{E}^3})dt, \tag{64}$$

where $\dot{x} := dx/dt$, $t \in \mathbb{R}$, being calculated on some time interval $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$, suitably related with the proper motion of the charged point particle ξ on the physically true time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the proper reference frame $\mathcal{K}_r$ sticked at the point charged particle and whose charge value is assumed so negligible that it exerts no influence on the external electromagnetic field. The problem now arises: how to compute correctly the variation $\delta S^{(\tau)} = 0$ of the action functional (64)?

To reply to this question we will turn to the Feynman reasonings from [26, 27], where he argued, when deriving the relativistic Lorentz force expression, that the real charged particle dynamics can be physically not ambiguously determined only with respect to the proper reference frame time parameter. Namely, Feynman wrote: "...we calculate a growth Δx for a small time interval Δt. But in the other reference frame the interval Δt may correspond to changing both t' and x', thereby at the change of the only t' the suitable change of x will be other... Making use of the quantity $d\tau$ one can determine a good differential operator $d/d\tau$, as it is invariant with respect to the Lorentz reference frames transformations". This means that if our charged particle ξ moves in the Minkowski space M^4 during the time interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference frame $\mathcal{K}$, its proper real and invariant time of motion with respect to the proper reference frame $\mathcal{K}_r$ will be respectively $[\tau_1, \tau_2] \subset \mathbb{R}$.

Observation 4.1. *All that above, in particular, means, having taken into account that the measurable electromagnetic four-potential $\mathcal{A} : M^4 \to T^*(M^4)$ has sense only with respect to the laboratory reference frame $\mathcal{K}$ with coordinates $(t, r) \in M^4$, that a physically reasonable and* ***relativistic invariant*** *action functional for a real charged point particle ξ motion should be initially constructed by means of an expression strongly calculated within this laboratory reference frame $\mathcal{K}$ and later suitably transformed subject to the proper reference frame $\mathcal{K}_r$ with coordinates $(\tau, r) \in \mathbb{E}^4$.*

As a corollary of the Feynman reasonings, we arrive at the necessity to rewrite the action functional (64) as

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} (-\xi\langle \mathcal{A}, \dot{x}\rangle_{M^4}) d\tau, \quad \delta x(\tau_1) = 0 = \delta x(\tau_2), \tag{65}$$

where $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$, under the additional constraint

$$\langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} = 1, \tag{66}$$

being equivalent to the infinitesimal transformation (56). Simultaneously the proper time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ is mapped on the time interval $[t_1, t_2] \subset \mathbb{R}$ by means of the infinitesimal transformation

$$dt = d\tau\sqrt{1 + |\dot{r}|^2}, \tag{67}$$

where $\dot{r} := dr/d\tau$, $\tau \in \mathbb{R}$. Thus, we can now pose the true least action problem equivalent to (65) as $\delta S^{(\tau)} = 0$, $\delta r(\tau_1) = 0 = \delta r(\tau_2)$, where the functional $S^{(\tau)} = \int_{\tau_1}^{\tau_2} [-\bar{W}\sqrt{1 + |\dot{r}|^2} + \xi\langle A, \dot{r}\rangle_{\mathbb{E}^3}] d\tau$ is characterized by the Lagrangian function $\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi\langle A, \dot{r}\rangle_{\mathbb{E}^3}$. Here we denoted, for further convenience, $\bar{W} := \xi\varphi$, being the suitable vacuum field [53, 54, 55, 57] potential function. The resulting Euler equation gives rise to the following relationships

$$\begin{aligned} P &:= \partial\mathcal{L}^{(\tau)}/\partial\dot{r} = -\bar{W}\dot{r}/\sqrt{1 + |\dot{r}|^2} + \xi A, \\ dP/d\tau &:= \partial\mathcal{L}^{(\tau)}/\partial r = -\nabla\bar{W}\sqrt{1 + |\dot{r}|^2} + \xi\nabla\langle A, \dot{r}\rangle_{\mathbb{E}^3}. \end{aligned} \tag{68}$$

Making now use once more of the infinitesimal transformation (67) and the crucial dynamical particle mass definition [53, 55, 57] (in the light speed units) $m := -\bar{W}$, we can easily rewrite equations (68) with respect to the parameter $t \in \mathbb{R}$ as the classical relativistic Lorentz force:

$$dp/dt = \xi E + \xi u \times B, \tag{69}$$

where we denoted

$$p := mu, \quad u := dr/dt, \quad B := \nabla \times A, \quad E := -\xi^{-1}\nabla\bar{W} - \partial A/\partial t.$$

Thus, we obtained once more the *relativistic* Lorentz force expression (69), but strongly different from (58), since the classical relativistic momentum expression of (59) does not completely coincide with our modified relativistic momentum expression

$$p = -\bar{W}u, \tag{70}$$

depending strongly on the scalar vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$. Yet, if to recall here that our action functional (65) was written under the assumption that the particle charge value ξ is negligible and exerting no essential influence on the electromagnetic field source, generated by external charged particles in rest, we can then put the vector potential $A = 0$ and make use of the before obtained in [12, 53, 57] result, that the vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$, owing to (69)-(70), approximately satisfies as $\xi \to 0$ the dynamical equation $d(-\bar{W}u)/dt = -\nabla\bar{W}$, whose solution will be exactly the expression

$$-\bar{W} = m_0/\sqrt{1-|u|^2}, \quad m_0 = -\bar{W}\big|_{u=0}. \tag{71}$$

Thereby, we have arrived, owing to (71) and (70), at the almost full coincidence of our result (69) for the relativistic Lorentz force with that of (58) under the condition $\xi \to 0$.

The obtained above results and inferences we will formulate as the following proposition.

Proposition 4.2. *Under the assumption of the negligible influence of a charged point particle ξ on an external electromagnetic field source a true physically reasonable **relativistic** invariant action functional can be given by expression (64), being equivalently defined with respect to the proper reference frame $\mathcal{K}_r$ in the form (65),(66). The resulting **relativistic** invariant Lorentz force (69) coincides in its form almost exactly with that of (58), obtained from the classical Einstein type relativistic invariant action functional (53), yet the momentum expression (70) strongly differs from the classical expression (59), taking into account the related vacuum field potential interaction energy impact.*

As an important corollary we make the following.

Corollary 4.3. *The Lorentz force expression (69) should be in due course corrected in the case when the weak charge ξ influence assumption made above does not hold. Moreover, its physically reasonable derivation should be grounded on the relativistic invariant least action functional (65).*

Remark 4.4. Concerning the infinitesimal relationship (67) one can observe that it reflects the Euclidean nature of the transformations $\mathbb{R} \ni t \rightleftharpoons \tau \in \mathbb{R}$.

In spite of the results obtained above by means of two different least action principles (53) and (65), we must claim here that the first one possesses some serious logical controversies, which may give rise to unpredictable, unexplainable and even nonphysical effects. Amongst these controversies we mention:

i) the definition of Lagrangian function (55) as an expression, depending on the external and undefined rest mass parameter with respect to the rest reference frame $\mathcal{K}_r$ time $\tau \in \mathbb{R}$, but serving as an variational integrand with respect to the laboratory reference frame $\mathcal{K}$ time $t \in \mathbb{R}$;
ii) the least action condition (53) is calculated with respect to the fixed boundary conditions at the ends of a time interval $[t_1, t_2] \subset \mathbb{R}$, thereby the resulting dynamics becomes strongly dependent on the chosen laboratory reference frame $\mathcal{K}$, what is, following the Feynman arguments [26, 27], physically unreasonable;
iii) the resulting relativistic particle mass and its energy depend only on the particle velocity in the laboratory reference frame $\mathcal{K}$, not taking into account the present vacuum field potential energy, exerting not trivial action on the particle motion;
iv) the assumption concerning the negligible influence of a charged point particle on the external electromagnetic field source is also physically inconsistent.

Thus, we can get strongly convinced that the dual approach to formulating the relativistic least action principle for a point charged particle dynamics, strongly based on the deep classical Amper's and Feynman's physical reasonings, is suitable for real modern physics applications absolutely supporting the relativistic invariance doctrine. Otherwise, those classical Amper's and Feynman's physical reasonings make it possible to reformulate the least action principle for a point charged particle dynamics, giving rise to the modified Lorentz force expression, approved in many physical experiments.

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REFERENCES

1. M. Abraham, *Dynamik des Electrons. Nachrichten von der Geselschaft der Wissenschafften zu Göttingen*, Mathematisch-Physikalische Klasse, 1902, S. 20.
2. R. Abraham, J. Marsden, *Foundations of Mechanics*, Benjamin Cummings, NY, 1978.
3. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, NY, 1978.
4. B.M. Barbashov, V.V.Nesterenko, *Introduction to the relativistic string theory*, World Scientific, Singapore, 1990.
5. B.M. Barbashov, V.N. Pervushin, A.F. Zakharov, V.A. Zinchuk, *The Hamiltonian approach to general relativity and CNB-primordal spectrum*, 2006 (arXiv:hep-th/0606054).
6. T.W. Barrett, *Electromagnetic phenomena not explained by Maxw ell's equations,* in: A. Lakhtakia (ed.), Essays on the Formal Aspects of Maxwell Theory (World Scientific, 1993), pp. 8–86.
7. R. Becker, *Theorie der Elektrizitat*, Bd. II., Elektronentheorie. Berlin: B.G. Teubner, 1933.
8. D. Blackmore, A.K. Prykarpatsky, V.Hr. Samoylenko, *Nonlinear dynamical systems of mathematical physics: spectral and differential-geometrical integrability analysis*, World Scientific Publ., NY, USA, 2011.
9. D. Blackmore, A.K. Prykarpatsky, *The Analysis of Lagrangian and Hamiltonian Properties of the Classical Relativistic Electrodynamics Models and Their Quantization*, Found Phys. **40** (2010) 469–493.
10. N.N. Bogolubov, N.N. Bogolubov (Jr.), *Introduction into quantum statistical mechanics*, World Scientific, NJ, 1986.
11. N.N. Bogolubov (Jr.), A.K. Prykarpatsky, *The Maxwell Electromagnetic Equations and the Lorentz Type Force Derivation - The Feynman Approach Legacy*, Int. J. Theor. Phys. **5** (2012) 237–245.
12. N.N. Bogolubov (Jr.), A.K. Prykarpatsky, *Quantum method of generating Bogolubov functionals in statistical physics: current Lie algebras, their representations and functional equations*, Physics of Elementary Particles and Atomique Nucleus, **17**:4 (1986) 791–827.
13. N.N. Bogolubov (Jr.), A.K. Prykarpatsky, U. Taneri, Y.A. Prykarpatsky, *The electromagnetic Lorentz condition problem and symplectic properties of Maxwell- and Yang–Mills-type dynamical systems*, J. Phys. A: Math. Theor. **42** (2009) 165401 (16pp).
14. N.N. Bogolubov, D.V. Shirkov, *Introduction to quantized field theory,* Moscow, Nauka, 1984 (in Russian)
15. B.M. Bolotovskii, S.N. Stolyarov, *Radiation from and energy loss by charged particles in moving media*, Sov. Phys. Usp. **35**:2 (1992) 143–150.
16. C.H. Brans, R.H. Dicke, *Mach's principle and a relativistic theory of gravitation*, *Phys. Rev*. **124** (1961), 925.
17. L. Brillouin, *Relativity reexamined*, Academic Press Publ., NY and London, 1970.
18. S. Deser, R. Jackiw, *Time travel?*, 1992 (arXiv:hep-th/9206094).
19. B. Di Bartolo, *Classical theory of electromagnetism*, World Sci., NY, 2004.
20. P.A.M. Dirac, *An extensible model of the electron,* Proc. Roy. Soc. Ser. A, **167** (1938) 148–169.
21. B.A. Dubrovin, S.P. Novikov, A.T. Fomenko, *Modern geometry,* Nauka, Moscow, 1986.
22. F.J. Dyson, *Feynman's proof of the Maxwell equations*, Am. J. Phys. **58** (1990) 209–211.
23. F.J. Dyson, *Feynman at Cornell*, Phys. Today **42**:2 (1989) 32–38.
24. A. Einstein, *Jahrbuch Radioaktivitat*, V, 1907, p.422.
25. L.D. Faddeev, *Energy problem in the Einstein gravity theory*, Russian Physical Surveys, **136**:3 (1982) 435–457 (in Russian)

26. R. Feynman, R. Leighton, M. Sands, *The Feynman Lectures on Physics. Electrodynamics*, v. 2, Addison-Wesley, Publ. Co., Massachusetts, 1964.

27. R. Feynman, R. Leightonm, M. Sands, *The Feynman lectures on physics. The modern science on the Nature. Mechanics. Space, time, motion.* v. 1, *Addison-Wesley*, Publ. Co., Massachusetts, 1963.

28. T.L. Gill, W.W. Zachary, *Two mathematically equivalent versions of Maxwell equations*, Preprint, University of Maryland, 2008.

29. T.L. Gill, W.W. Zachary, *Two mathematically equivalent versions of Maxwell's equations*, Found. Physics. **4** (2011) 99–128.

30. T.L. Gill, W.W. Zachary, J. Lindsey, *The classical electron problem.* Foundations of Physics, **31**:9 (2001) 1299–1355.

31. C. Godbillon, *Geometrie Differentielle et Mecanique Analytique*, Hermann Publ., Paris, 1969.

32. O.Ye. Hentosh, M.M. Prytula, A.K. Prykarpatsky, *Differential-geometric integrability fundamentals of nonlinear dynamical systems on functional menifolds*, (The second revised edition), Lviv University Publisher, Lviv, Ukraine, 2006.

33. K. Huang, *Statistical Mechanics*, John Wiley & Sons, 1987.

34. R. Jackiw, A.P. Polychronakos, *Dynamical Poincare symmetry realized by field-dependent diffeomorhisms,* 1998 (arXiv:hep-th/9809123).

35. R. Jackiw, *Lorentz violation in a diffeomorphism-invariant theory*, 2007 (arXiv:hep-th/0709.2348).

36. J.D. Jackson, *Classical electrodynamics.* 3-rd Edition., NY, Wiley, 1999.

37. H. Kleinert, *Path integrals*, World Sci. Second Edition, 1995, p.685.

38. I.A. Klymyshyn, *Relativistic astronomy*, Kyiv, "Naukova Dumka" Publ., 1980

39. L.D. Landau, E.M. Lifshitz, *Field theory.* v. 2, "*Nauka*" Publ., Moscow, 1973.

40. H.A. Lorentz, *Electromagnetic phenomena in a system moving with any velocity smaller than that of light*, Proceedings of the Royal Netherlands Academy of Arts and Sciences, **6** (1904) 809–831.

41. A.A. Martins, M.J. Pinheiro, *On the electromagnetic origin of inertia and inertial mass.* Int. J. Theor. Phys. **47** (2008) 2706–2715.

42. R. Medina, *Radiation reaction of a classical quasi-rigid extended particle*, J. Phys. A: Math. Gen., (2006) 3801–3816.

43. N.D. Mermin, *Relativity without light*, Am. J. Phys. **52** (1984) 119–124.

44. H. Minkowski, *Raum und Zeit*, Physik. Z. **10** (1909) p.104.

45. V.B. Morozov, *On the question of the electromagnetic momentum of a charged body*, Physics Uspekhi, **181**:4 (2011) 389–392.

46. R.P. Newman, *The Global Structure of Simple Space-Times,*Comm. Mathem. Physics, **123** (1989) 17–52.

47. A. Ori, E.J. Rosenthal, *Calculation of the Self Force using the Extended-Object Approach*, Math. Phys. **45** (2004) 2347.

48. A. Ori, E.J. Rosenthal, *Universal self-force from an extended object approach,* Phys. Rev. D, **68** (2003) 041701R.

49. W. Pauli, *Theory of Relativity*. Oxford Publ., 1958.

50. H. Poincare, *Sur la Dynamique de l'Electron*, C.R. Acad. Sci. (Paris) **140** (1905) p.1504.

51. A.K. Prykarpatski, *Classical Electromagnetic Theory Revisiting: The A.M. Ampere Law and the Vacuum Field Theory Approach*, Universal J. Phys. Appl. **2**:8 (2014) 381–413.

52. A.K. Prykarpatski, *The classical Abraham-Lorentz electron mass theory legacy. Part I: The Amper law aspects revisiting*, Math. Bull. T. Shevchenko Sci. Soc. **12** (2015), 72–108.

53. A.K. Prykarpatsky, N.N. Bogolubov (Jr.), U. Taneri, *The vacuum structure, special relativity and quantum mechanics revisited: a field theory no-geometry approach*, Theor. Math. Phys. **160**:2 (2009), 1079–1095; (arXiv lanl: 0807.3691v.8 [gr-gc] 24.08.2008)
54. A.K. Prykarpatsky, N.N. Bogolubov (Jr.), U. Taneri, *The field structure of vacuum, Maxwell equations and relativity theory aspects*, Preprint ICTP, Trieste, IC/2008/051.
55. A.K. Prykarpatsky, N.N. Bogolubov (Jr.), U. Taneri, *The Relativistic Electrodynamics Least Action Principles Revisited: New Charged Point Particle and Hadronic String Models Analysis*, Int. J. Theor. Phys. **49** (2010) 798–820.
56. H.E. Puthoff, *Casimir vacuum energy and the semicalssical electron*, Int. J. Theor. Phys. **46** (2007) 3005-3008.
57. O. Repchenko, *Field physics*. Moscow, Galeria Publ., 2005 (in Russian).
58. F. Rohrlich, *Classical Charged Particles*, Reading, Mass.: Addison-Wesley, 1965.
59. G. Rousseaux, *The gauge non-invariance of classical electromagnetism*, 2005 (arXiv:physics/0506203v1).
60. W. Thirring, *Classical Mathematical Physics*, Springer, Third Edition, 1992.
61. R. Weinstock, *New approach to special relativity*, Am. J. Phys. **33** (1965) 640–645.

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