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A NOTE ON ULTRAFILTERS ON BOOLEAN ALGEBRAS

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Under some conditions on the action of a semigroup S on a Boolean algebra B, we consider the natural action of S on the Stone space $Ult(B)$ of B and characterize minimal closed S-invariant subsets of $Ult(B)$. As a corollary, we get: if $1_B = a_1 \vee \cdots \vee a_n$ then some element a_i is relatively large with respect to the action of S.

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При деяких обмеженнях на дію напівгрупи S на булевій алгебрі B, розглянуто природну дію S на просторі Стоуна $Ult(B)$ алгебри B і охарактеризовано мінімальні замкнені S-інваріантні підмножини в $Ult(B)$. Як наслідок, отримано, що для довільного покриття $1_B = a_1 \vee \cdots \vee a_n$ один з елементів a_i є у певному сенсі великим відносно дії напівгрупи S.

1. Introduction

Let X be a set endowed with the action $(S, X) \to X$, $(s, x) \mapsto sx$, of a semigroup S (i.e. $(st)x = s(tx)$ for all $s, t \in S$, $x \in X$). We denote by $\mathcal{P}(X)$ the Boolean algebra of all subsets of X and extend the action of S to $\mathcal{P}(X)$ by the rule $A \mapsto sA$, $sA = \{sa : a \in A\}$.

We endow X with the discrete topology, consider the Stone-Čech compactification βX of X, identify βX with the set of all ultrafilters on X and define the action of S on βX by the rule $p \mapsto sp$, $sp = \{Y \subseteq X : s^{-1}Y \in p\}$, $s^{-1}Y = \{x \in X : sx \in Y\}$. Then, for each $s \in S$, the mapping $\beta X \to \beta X$, $p \mapsto sp$, is continuous.

A subset L of βX is called S-*invariant* if $sp \in S$ for all $s \in S$ and $p \in L$. By Zorn's lemma and compactness of βX, each closed S-invariant subset of βX contains some minimal under inclusion closed S-invariant subset. In the case of the left regular S-space ($X = S$ and sx is the product of s and x in S), minimal closed S-invariant

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subsets are characterized in [6, Section 4.4]. It should be mentioned that ultrafilters on S which belong to some minimal closed S-invariant subset of βS have vast applications in Ramsey Theory [6, Chapter 14].

In this note, our goal is to describe minimal closed S-invariant subsets in the Stone space of a Boolean algebra endowed with an action of a semigroup S.

Let B be a Boolean algebra with the minimal element $\mathbf{0}$ and the maximal element $\mathbf{1}$. We recall that a subset Φ of B is a *filter* if $\mathbf{1} \in \Phi$, $\mathbf{0} \notin \Phi$ and

$$x, y \in \Phi,\ x \leq z \Rightarrow x \wedge y \in \Phi,\ z \in \Phi.$$

The set of all filters on B is partially ordered by inclusion. A maximal filter Φ in this ordering is called an *ultrafilter*. We use the characteristic property of ultrafilters: a filter Φ is an ultrafilter if and only if, for every $x \in B$, either $x \in B$ or its complement $\overline{x} \in B$.

We identify the subsets of B with the elements of $\{0, 1\}^B$, endow $\{0, 1\}$ with the discrete topology and $\{0, 1\}^B$ with the product topology. Then the set $Ult(B)$ of all ultrafilters on B is closed in $\{0, 1\}^B$. The space $Ult(B)$ is called the *Stone space* of B (see [10, Chapter 1]). We note that the subsets $\{p \in Ult(B) : a \in p\}$, $a \in B$, form a base for open sets in $Ult(B)$.

We suppose that a Boolean algebra B is endowed with an action $S \times B \to B$, $(s, b) \to sb$, of a semigroup S such that for every $s \in S$, following conditions are satisfied

(1) $\forall x \in B\ (sx = \mathbf{0} \Leftrightarrow x = \mathbf{0})$;
(2) $\forall x, y \in B\ (x \leq y \Rightarrow sx \leq sy)$;
(3) $\forall x \in B\ \exists y \in B\ (sy \leq x \& s\overline{y} \leq \overline{x})$.

Given any $s \in S$ and $p \in Ult(B)$, we set

$$sp = \{a \in B : \exists x \in p\ (sx \leq a)\}$$

By (1) and (2), sp is a filter. By (3), sp is an ultrafilter. Moreover, $S \times Ult(B) \to Ult(B)$, $(s, p) \mapsto sp$, is an action and, for each $s \in S$ the mapping $Ult(B) \to Ult(B)$, $p \mapsto sp$, is continuous. Hence, each closed S-invariant subset of $Ult(B)$ contains some minimal closed S-invariant subset.

F. Wehrung pointed out that the condition (1)–(3) are equivalent to the *weak distributivity* of s: for every $c \in B$, if $sc \leq a \vee b$ then there is a decomposition $c = x \vee y$ such that $sx \leq a$ and $sy \leq b$.

For weakly distributive mappings, see [5]. Each weakly distributive s respects join: $s(x \vee y) = sx \vee sy$.

2. Results

We suppose that a Boolean algebra B is endowed with an action of a semigroup S such that each $s \in S$ is weakly distributive.

For every $a \in B$ and $p \in Ult(B)$ we denote

$$\Delta_p(a) = \{s \in S : a \in sp\}.$$

If a semigroup S acts on a compact space T so that, for each $s \in S$, the mapping $T \to T$, $x \mapsto sx$, is continuous then, by the Birkhoff theorem (see [3], [4]), a point $p \in T$ belongs to some minimal closed S-invariant subset of T if and only if p is *uniformly recurrent*, i.e. for every neighborhood U of p, there exists a finite subset F of S such that, for every $s \in S$, there exists $f \in F$ such that $fs \in \{t \in S : tp \in U\}$.

Applying this theorem to the pair $(S, Ult(B))$, we get

Theorem 2.1. *An element $p \in Ult(B)$ belongs to some minimal closed S-invariant subset M of $Ult(B)$ if and only if for every $a \in p$ there exists a finite subset F of S such that $S = F^{-1}\Delta_p(a)$.*

Given $a \in B$, how can one detect whether a is an element of some uniformly recurrent point $p \in Ult(B)$? To answer this question, we suppose that, for any $s \in S$

(4) $\forall x \in B\ \exists y \in B\ (sy \leq x\ \&\ (\forall z \in B\ (sz \leq x \Rightarrow z \leq y)))$.

G. Bergman noticed that (4) follows from the weak distributivity of s: since $\mathbf{1} = x \vee \overline{x}$, there is y such that $sy \leq x$, $s\overline{y} \leq \overline{x}$. If $z \wedge \overline{y} \neq \varnothing$ then $sz \wedge \overline{x} \neq \varnothing$ so y satisfies (4).

We note that, for every $x \in B$, there is unique $y \in B$ satisfying (4) and put $y = s^{-1}x$. We use the following simple observation: $\Delta_p(a) = \{s \in S : s^{-1}a \in p\}$.

We say that an element $a \in B$ is

- *large* if there exist $s_1, \dots, s_n \in S$ such that $s_1^{-1}a \vee \dots \vee s_n^{-1}a = \mathbf{1}$;
- *thick* if $\overline{a}$ is not large;
- *prethick* if there exist $s_1, \dots, s_n \in S$ such that $s_1^{-1}a \vee \dots \vee s_n^{-1}a$ is thick.

Theorem 2.2. *For $a \in B$, the following statements hold*

(i) a is large if and only if $\Delta_p(a) \neq \varnothing$ for every $p \in Ult(B)$;
(ii) a is thick if and only if there exists $p \in Ult(B)$ such that $\Delta_p(a) = S$.

Proof. (i) If A is large then we choose $s_1, \dots, s_n \in S$ such that $\mathbf{1} = s_1^{-1}a \vee \dots \vee s_n^{-1}a$. Since p is an ultrafilter, $s^{-1}a \in p$ for some $i \in \{1, \dots, n\}$ so $a \in sp$, $sp \in \Delta(a)$ and $\Delta(a) \neq \varnothing$.

We assume that a is not large. Then the family

$$\{\overline{s_1^{-1}a} \wedge \dots \wedge \overline{s_n^{-1}a} : s_1, \dots, s_n \in S, n \in \mathbb{N}\}$$

is the base for some filter Φ on B. We take an arbitrary ultrafilter $p \in Ult(B)$ such that $\Phi \subseteq p$. Then $\overline{s_1^{-1}a} \in p$ for each $s \in S$ so $s^{-1}a \notin p$ and $\Delta_p(a) = \varnothing$.

(ii) follows from (i) and the definition of thick subsets. □

Theorem 2.3. *An element $a \in B$ is prethick if and only if there exists a uniformly recurrent point $p \in Ult(B)$ such that $a \in p$.*

Proof. We suppose that there is a uniformly recurrent point $p \in Ult(B)$ such that $a \in p$. By Theorem 2.1, there exists a finite subset F of S such that $S = F^{-1}\Delta_p(a)$.

We observe that $F^{-1}\Delta_p(a) = \Delta_p(\bigvee_{f\in F} f^{-1}a)$. By Theorem 2.2$(ii)$, $\bigvee_{f\in F} f^{-1}a$ is thick so a is prethick.

To prove the converse statement, we assume that a is prethick and find $s_1, \dots, s_n \in S$ such that $s_1^{-1}a \vee \dots \vee s_n^{-1}a$ is thick. By Theorem 2.2(ii), there is $q \in Ult(B)$ such that $s_1^{-1}a \vee \dots \vee s_n^{-1}a = sq$ for each $s \in S$. The S-invariant subset $cl(Sq)$ contains some minimal closed S-invariant subset M. We take an arbitrary $r \in M$ and pick $i \in \{1, \dots, n\}$ such that $s_i^{-1}a \in r$. Then $a \in sr$ and we put $p = sr$. □

Corollary 2.4. *If $a_1, \dots, a_n \in B$ and $a_1 \vee \dots \vee a_n = \mathbf{1}$ then there is $i \in \{1, \dots, n\}$ such that a_i is prethick.*

For $a \in B$, we denote $\Delta(a) = \{s \in S : s^{-1}a \wedge a \neq \mathbf{0}\}$ and observe that $\Delta(a) = \bigcup\{\Delta_p(a) : p \in Ult(B), a \in p\}$.

Corollary 2.5. *If $a_1, \dots, a_n \in B$ and $a_1 \vee \dots \vee a_n = \mathbf{1}$ then there is $i \in \{1, \dots, n\}$ and a finite subset F of S such that $S = F^{-1}\Delta(a_i)$.*

Corollary 2.5 could be strengthened (see comments 3, 4): there is F such that $|F| \leq 2^{2^{n-1}-1}$.

3. Comments

1. Let X be a set endowed with the action of a semigroup S. We put $B = \mathcal{P}(X)$, and note that the natural action of S on B satisfies (1), (2), (3). By [9], $A \in \mathcal{P}(X)$ is thick if and only if, for every finite subset K of X, there exists $s \in S$ such that $sK \subseteq A$. In the case of the left regular S-space, prethick elements of $\mathcal{P}(X)$ are called piecewise syndetic [6, p.101] so Theorems 4.39 and 4.40 from [6] are partial cases of Theorems 2.1 and 2.3. Another partial case: each element of S acts on a Boolean algebra B as an automorphism.

2. In 1995, the fist author asked the following question [7, Problem 13.44]: given a group G and an n-partition $\mathcal{P}$ of G, $n \in \mathbb{N}$, do there exit $A \in \mathcal{P}$ and a subset F of G such that $G = FAA^{-1}$ and $|F| \leq n$. Clearly, $AA^{-1} = \Delta(A)$.

For current state of this open problem see [1], [2]. We mention only that an answer is positive if either G is amenable, or $n \leq 3$, or $x^{-1}Ax = A$ for any $A \in \mathcal{P}$ and $x \in G$. If G is an arbitrary group then there are $A, B \in \mathcal{P}$ and subsets F, H of G such that $G = FAA^{-1}$, $|F| \leq n!$ and $G = HBB^{-1}B$, $|H| \leq n$.

3. We generalize above question to semigroup and ask

(5) given an n-partition $\mathcal{P}$ of a semigroup S, do there exist $A \in \mathcal{P}$ and a subset F of S such that $S = F^{-1}\Delta(A)$, $|F| \leq n$?

By [8, Theorem 1], there is F with $|F| \leq 2^{2^{n-1}-1}$ so (5) has positive answer if $n = 2$. G. Bergman proved (5) for $n = 3$. By [8, Theorem 2], an answer to (5) is affirmative for each finite semigroup S.

4. We suppose that a semigroup S acts on a Boolean algebra B weakly distributively. If $\mathbf{1}_B = a_1 \vee \cdots \vee a_n$, do there exist a_i and a finite subset F of S such that $S = F^{-1}\Delta(a_i)$.

We show that this question can be reduced to (5). We identify B with the family of all clopen subsets of some compact extremely disconnected Hausdorff space X, take $s \in S$ and construct $f : X \to X$ such that $sU = f(U)$ for each $U \in B$. We take $x \in X$, note that, by compactness of X, $\bigcap\{sU : x \in U, U \in B\} \neq \varnothing$, take $y \in \bigcap\{sU : x \in U, U \in B\}$ and show that $\bigcap\{sU : x \in U, U \in B\} = \{y\}$. Let $z \in X$, $z \neq y$. We choose clopen subset V, W of X such that $y \in V$, $z \in W$ and $V \cap W \neq \varnothing$. Since s is weakly distributive, there are $U, U' \in B$ such that $x \in U$, $sU \subset V$, $sU' \subset W$ so $z \notin sU$. We put $f(x) = y$.

Now let $U_1, \ldots, U_n \in B$ and $X = U_1 \cup \cdots \cup U_n$. We fix some $t \in X$, and denote $V_i = \{s \in S : f(t) \in U_i\}$. In such a way, the Δ-question for $U_1, \ldots, U_n$ is reduced to the Δ-question for $V_1, \ldots, V_n$ because "partition" in (5) is equivalent to "covering".

REFERENCES

1. T. Banakh, I. Protasov, S. Slobodianiuk, *Densities, submeasures and partitions of G-spaces and groups*, Algebra Discrete Math. **17**:2 (2014), 193–221.
2. T. Banakh, O. Ravsky, S. Slobodianiuk, *On partitions of G-spaces and G-latices*, Intern. J. Algebra Computation **26**:2 (2016), 283–308.
3. G. Birkhoff, *Dynamical systems*, Amer. Math. Soc. Coll. Publ. **9** (1927).
4. R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.
5. G. Grätzer, F. Wehrung (eds), *Lattice Theory: special topics and applications*, Volume **1**, Birkhäuser, 2014.
6. N. Hindman, D. Strauss, *Algebra in the Stone-Čech compactification*, 2nd edition, de Gruyter, 2012.
7. V.D. Mazurov, E.I. Khukhro (eds), *Unsolved problem in group theory, the Kourovka notebook*, 13-th augmented edition, Novosibirsk, 1995.
8. I. Protasov, K. Protasova, *A note-question on partitions of semigroups*, Mat. Stud. **44**:1 (2015), 104–106.
9. I.V. Protasov, S. Slobodianiuk, *On the subset combinatorics of G-spaces*, Algebra Discrete Math. **17**:1 (2014), 98–109.
10. R. Sikorski, *Boolean Algebras*, Springer, 1964.

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