GROUPS WITH NORMAL INFINITE CYCLIC SUBGROUPS

©2006 *Mykola CHERNIKOV*

Institute of Mathematics of NASU,
3 Tereshenkivska Str., Kyiv 01601, Ukraine

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The complete description of nonperiodic groups with normal infinite cyclic subgroups is obtained, and it is observed that such groups are just all nonperiodic groups with the minimal condition for nonnormal cyclic subgroups.

The complete description of finite groups with normal cyclic subgroups was obtained in 1897 by R.Dedekind [1]. Further, the complete description of arbitrary groups in which all cyclic subgroups are normal was obtained in 1933 by R.Baer [2]. According to R.Baer's Theorem [2], nonperiodic groups with normal cyclic subgroups are just nonperiodic abelian groups. Simple examples show that nonperiodic groups with normal infinite cyclic subgroups are not necessarily abelian. The following Theorem, which is the main result of the present paper, completely describes nonperiodic groups with normal infinite cyclic subgroups and establishes that the class of all such groups is just the class of all groups in which almost all infinite cyclic subgroups are normal. Moreover, by Theorem, a nonperiodic group satisfies the minimal condition for nonnormal cyclic subgroups iff all its infinite cyclic subgroups are normal. Note that the complete description of infinite groups in which almost all cyclic subgroups are normal was obtained by the author of the present paper in [3].

Below $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. All other notations used in the present paper are standard.

Theorem. *Let G be a nonperiodic group. Then the following statements are equivalent:*

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(i) *All infinite cyclic subgroups of G are normal in it.*
(ii) *Almost all infinite cyclic subgroups of G are normal in it.*
(iii) *Almost all subgroups of each cyclic subgroup of G are normal in G.*
(iv) *G satisfies the minimal condition for nonnormal cyclic subgroups.*
(v) *All torsion-free subgroups of G are normal in it.*
(vi) *Either G is abelian, or for some normal abelian subgroup A of G and some element b of G the following hold:*

$$|G : A| = 2, \tag{1}$$

$$G = A\langle b\rangle, \tag{2}$$

$$a^b = a^{-1}, \tag{3}$$

where a is any element of A, and

$$b^4 = 1. \tag{4}$$

Further, if in the statement (vi) *G is not abelian, then for any $u \in G\backslash A$,*

$$u^2 = b^2 \tag{5}$$

and

$$|\langle u\rangle| = |\langle b\rangle|, \tag{6}$$

and

$$Z(G) = \{a : a \in A \text{ and } a^2 = 1\}. \tag{7}$$

Remark 1. Remind that by definition a group satisfies the minimal condition for certain subgroups if it has no infinite strictly descending chains of these subgroups.

Remark 2. In connection with the statement (iv) of Theorem, note that the minimal condition for nonnormal cyclic subgroups is, obviously, equivalent to the minimal condition for nonnormal infinite cyclic subgroups, and, clearly, all periodic groups satisfy this condition.

Preface the proof of Theorem with the following Proposition.

Lemma 1. *Let G be a nonperiodic group in which all infinite cyclic subgroups are normal. Then for every infinite cyclic subgroups $\langle g\rangle$ and $\langle h\rangle$, the following relation is fulfilled*

$$C_G(\langle g\rangle) = C_G(\langle h\rangle). \tag{8}$$

Preface the proof of Lemma 1 with the following remark.

Remark 3. Remind the following. For an infinite cyclic subgroup $\langle v \rangle$ of the group G the relations

$$Z(G) \not\supseteq \langle v \rangle \trianglelefteq G$$

imply the relations

$$|G : C_G(\langle v \rangle)| = 2$$

and

$$v^b = v^{-1} \tag{9}$$

where b is any element from $G \backslash C_G(\langle v \rangle)$.

Proof of Lemma 1. Assume that (8) is false. Let, for instance,

$$C_G(\langle g \rangle) \backslash C_G(\langle h \rangle) \neq \varnothing.$$

Take $b \in C_G(\langle g \rangle) \backslash C_G(\langle h \rangle)$. Then for any $u \in \langle g \rangle$ and $v \in \langle h \rangle$,

$$u^b = u \text{ and } v^b = v^{-1} \tag{10}$$

(see (9)). In view of (10), for $w \in \langle g \rangle \cap \langle h \rangle$, $w^b = w = w^{-1}$, i.e. $w^2 = 1$. Consequently, $w = 1$. Thus

$$\langle g \rangle \cap \langle h \rangle = 1. \tag{11}$$

Since

$$\langle g \rangle, \langle h \rangle \trianglelefteq G, \tag{12}$$

with regard to (11),

$$[g, h] = 1. \tag{13}$$

Because of $\langle gh \rangle \trianglelefteq G$, by virtue of (12) and (13), either

$$(gh)^b = gh = g^b h^b = gh^{-1} \tag{14}$$

or

$$(gh)^b = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = g^b h^b = gh^{-1}. \tag{15}$$

Thus, with regard to (14) and (15), either $gh = gh^{-1}$ or $g^{-1}h^{-1} = gh^{-1}$. Consequently, $h = h^{-1}$ or $g^{-1} = g$, which is a contradiction. So (8) is valid. Lemma is proven.

Proof of Theorem. Obviously, (i) $\to$ (ii) $\to$ (iii) $\to$ (iv) and (v) $\to$ (i).

(i) $\to$ (v). Indeed, any nonidentity torsion-free subgroup of G is covered by its infinite cyclic subgroups and, at the same time, is normal in G.

Thus (i) $\leftrightarrow$ (v).

Take any infinite cyclic subgroup $\langle g\rangle$ of G.

(iv) $\to$ (i). Since $\langle g^{2^k}\rangle \supsetneqq \langle g^{2^{k+1}}\rangle$, $k \in \mathbb{Z}^+$, and $\langle g^{3^k}\rangle \supsetneqq \langle g^{3^{k+1}}\rangle$, $k \in \mathbb{Z}^+$, for some $j, l \in \mathbb{Z}^+$, we have

$$\langle g^{2^j}\rangle, \langle g^{3^l}\rangle \trianglelefteq G. \tag{16}$$

Further, there exist $m, n \in \mathbb{Z}$, such that $2^j m + 3^l n = 1$. Then

$$\langle g\rangle = \langle g^{2^j m}\rangle\langle g^{3^l n}\rangle. \tag{17}$$

In view of (16),

$$\langle g^{2^j m}\rangle, \langle g^{3^l n}\rangle \trianglelefteq G. \tag{18}$$

So, with regard to (17) and (18), $\langle g\rangle \trianglelefteq G$.

Thus (i) $\leftrightarrow$ (ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv) $\leftrightarrow$ (v).

(i) $\to$ (vi). Let G be not abelian. Put $A = C_G(\langle g\rangle)$. Take any $a \in A$. If a is of infinite order, then, by Lemma 1,

$$C_G(\langle a\rangle) = A. \tag{19}$$

By virtue of (19), $a \in Z(A)$. Assume that a is of finite order. Since $g \in Z(A)$, we have

$$(ga)^{|\langle a\rangle|} = g^{|\langle a\rangle|}a^{|\langle a\rangle|} = g^{|\langle a\rangle|}. \tag{20}$$

Inasmuch as g is of infinite order, with regard to (20), ga is of infinite order too. Consequently, $ga \in Z(A)$ (see above). So $a = g^{-1}(ga) \in Z(A)$. Thus A is abelian. Therefore $A \neq G$. Consequently, (1) is true (see Remark 3).

Take $b \in G\backslash A$. Then, with regard to (1), (2) is correct.

If a is of infinite order, then, with regard to (19), (3) is fulfilled (see Remark 3). Let a be of finite order. Since g and ga are of infinite order, $g^b = g^{-1}$ and $(ga)^b = (ga)^{-1}$. So, because of A is abelian,

$$a^b = g^{-b}(ga)^b = g(ga)^{-1} = ga^{-1}g^{-1} = a^{-1}.$$

Thus (3) is valid.

In consequence of (1), $b^2 \in A$. Therefore, by proven above, $b^{2b} = b^{-2}$. So $b^{2b} = b^2 = b^{-2}$, i.e. $b^4 = 1$. Thus (4) is valid.

Further, if $u \in G\backslash A$, then for some $a \in A$, $u = ba$ (see (1) and (2)). Then, with regard to (3), $u^2 = b^2a^ba = b^2$. Thus (5) is correct. Therefore, obviously, (6) is valid.

Since A is abelian, $AZ(G)$ is abelian too. Consequently, $|G : AZ(G)| \neq 1$. Therefore, with regard to (1),

$$1 \neq |G : AZ(G)| \leq |G : A| = 2.$$

Consequently, $|G : AZ(G)| = |G : A|$, i.e.

$$Z(G) \subseteq A. \tag{21}$$

Take any $a \in Z(G)$. Then, with regard to (21),

$$a^b = a^{-1} = a \tag{22}$$

(see (3)), i.e. $a^2 = 1$.

If $a^2 = 1$ and $a \in A$, then, with regard to (3), (22) are fulfilled. Therefore, since A is abelian and (2) and (22) hold, $a \in Z(G)$. Thus (7) is valid.

(vi) $\rightarrow$ (i). If G is abelian, then (i) is valid. Let G be not abelian. Since all elements from $G \backslash A$ are of finite order (see (6) and (4)),

$$\langle g \rangle \subseteq A. \tag{23}$$

So $g^b = g^{-1}$ (see (3)), and, at the same time,

$$\langle b \rangle \subseteq N_G(\langle g \rangle). \tag{24}$$

Since A is abelian, with regard to (23),

$$A \subseteq N_G(\langle g \rangle). \tag{25}$$

In consequence of (2), (24) and (25), $\langle g \rangle \trianglelefteq G$.

Theorem is proven.

Remind that, by definition, IH-groups are just infinite nonabelian groups possessing infinite abelian subgroups, in which all such subgroups are normal (S.N. Chernikov; see, for instance, Definition 4.1 [4]).

Corollary 1 (S.N.Chernikov; see, for instance, [4], Theorem 4.6). *The nonabelian group G containing elements of infinite order is an IH-group iff its centre is finite and it has some abelian normal subgroup A of index 2 and some cyclic subgroups $\langle b \rangle$ of order 2 or 4, such that*

$$G = A\langle b \rangle \text{ and } a^b = a^{-1},$$

where a is any element of A.

An infinite nonabelian group satisfying the minimal condition for nonnormal abelian subgroups is called an I-group (S.N. Chernikov; see, for instance, [4], Definition 4.2). Clearly, the class of all IH-groups is a subclass of the class of all I-groups.

Corollary 2 (S.N. Chernikov; see, for instance, [4], Theorem 4.10). *Each nonperiodic I-group G is an IH-group.*

Proof of Corollaries 1 and 2. Take G from Corollary 2. Then G satisfies the minimal condition for nonnormal cyclic subgroups. Therefore, in view of Theorem, all infinite cyclic subgroups of G are normal in it, there exist A and $\langle b\rangle$ such as in Corollary 1 and $Z(G)$ is an elementary abelian 2-subgroup of A.

Suppose that $Z(G)$ is infinite. Since $\langle b\rangle Z(G)/\langle b\rangle$ is infinite elementary abelian, obviously, there exists some infinite descending chain

$$A_1/\langle b\rangle \supset A_2/\langle b\rangle \supset \ldots \supset A_k/\langle b\rangle \supset A_{k+1}/\langle b\rangle \supset \ldots$$

of subgroups of $\langle b\rangle Z(G)/\langle b\rangle$. Then for some $k \in \mathbb{N}$,

$$A_k \trianglelefteq G. \tag{26}$$

Take any infinite cyclic subgroup $\langle u\rangle$ of A. Then

$$\langle u\rangle \trianglelefteq G \tag{27}$$

and for each $a \in \langle u\rangle$, $a^b = a^{-1}$. So $\langle u\rangle \cap C_G(\langle b\rangle) = 1$. Hence, because of $b \in A_k$,

$$\langle u\rangle \cap A_k = 1. \tag{28}$$

In consequence of (26)–(28), $[\langle u\rangle, A_k] = 1$. At the same time, $[\langle u\rangle, \langle b\rangle] = 1$, which is a contradiction. Thus $Z(G)$ is finite.

Take any infinite abelian subgroup T of G. If $T \not\subseteq A$, then, because of $|G : A| = 2$, $G = TA$. Therefore $T \cap A \subseteq Z(G)$. But $T \cap A$ is infinite, which is a contradiction. Thus $T \subseteq A$.

Take any $a \in T$. Then, because of A is abelian and $a \in A$ and $G = A\langle b\rangle$, we have $G = C_G(\langle a\rangle)\langle b\rangle$. Also $\langle b\rangle \in N_G(\langle a\rangle)$. So $\langle a\rangle \trianglelefteq G$. Therefore $T \trianglelefteq G$. Thus G is an IH-group.

Corollaries are proven.

The next proposition is directly connected with Theorem (see the statement (vi)).

Lemma 2. *Let A be an abelian group and c be its element of order ≤ 2. Then there exists some group G such that*

$$\langle b\rangle \leq G, \quad A \lhd G = A\langle b\rangle \quad \text{and } A \cap \langle b\rangle = \langle c\rangle, \tag{29}$$

$$|G:A| = 2, \quad |\langle b\rangle| = 2|\langle c\rangle| \quad (= 4 \text{ or } 2) \tag{30}$$

and for any $a \in A$, (3) is fulfilled.

Further, if some group $G^* = A^*\langle b\rangle^*$ is the same as G above and also there exists some isomorphism φ of A onto A^* with $c^\varphi = c^*$, then $G \simeq G^*$. Moreover, the mapping: $ab^k \mapsto a^\varphi b^{*k}$, $a \in A$ and $k = 0, 1$, is an isomorphism of G onto G^*.

Proof. Let $F = A \leftthreetimes \langle u\rangle$ where $|\langle u\rangle| = 2|\langle c\rangle|$ and for every $a \in A$, $a^u = a^{-1}$. Put $G = F/\langle cu^2\rangle$ and $b = u\langle cu^2\rangle (\in G)$. Identify naturally $A\langle cu^2\rangle/\langle cu^2\rangle$ with A. It is easy to see that (29), (30) and (3) are fulfilled.

Proof of the last assertion of the present lemma is obvious.

Lemma is proven.

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ГРУПИ З НОРМАЛЬНИМИ НЕСКІНЧЕННИМИ ЦИКЛІЧНИМИ ПІДГРУПАМИ

Микола ЧЕРНІКОВ

Інститут математики НАН України,
вул. Терещенківська 3, Київ 01601

Отримано повний опис неперіодичних груп із нормальними нескінченними циклічними підгрупами і доведено, що всі такі групи є неперіодичними групами з умовою мінімальності для ненормальних циклічних підгруп.