ON A MEASURE OF ALGEBRAIC INDEPENDENCE OF VALUES OF JACOBI ELLIPTIC FUNCTIONS. I

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In the paper, an estimation of a measure of algebraic independence is proved for values of various algebraic points of the Jacobi elliptic function $\operatorname{sn}(z)$.

1. INTRODUCTION

Let $\wp(z)$ be the Weierstrass elliptic function with algebraic invariants g_2, g_3 and with complex multiplication over field $\mathbb{Q}(\tau)$, $\operatorname{sn}(z)$ be a Jacobi elliptic function, $\varkappa$ be an elliptic module $\operatorname{sn}(z)$, $\varkappa$ be an algebraic number, $\varkappa \neq 0; 1$. In [19,25], an elliptic analog of the Lindemann theorem is proved.

Theorem 1. *Let $\alpha_1, \dots, \alpha_k$ be algebraic numbers linearly independent over $\mathbb{Q}(\tau)$. Then the numbers $\wp(\alpha_1), \dots, \wp(\alpha_k)$ are algebraically independent over $\mathbb{Q}$.*

We refer to [3,23] for some information on arithmetic properties of numbers related to elliptic functions.

From relations between $\wp(z)$ and $\operatorname{sn}(z)$ and the proof of Theorem 1 in [16] an analog of the Lindemann theorem follows for $\operatorname{sn}(z)$.

Theorem 2. *If $\alpha_1, \dots, \alpha_n$ are algebraic numbers, linearly independent over $\mathbb{Q}$, then at least $[n/2]$ of $\operatorname{sn}(\alpha_1), \dots, \operatorname{sn}(\alpha_n)$ are algebraically independent over $\mathbb{Q}$.*

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An estimation of a measure of algebraic independence of values at algebraic points of an elliptic Weierstrass function is announced in [2,12,26], and also follows from published in [5] results. An estimation proved in [15, 16], is best of known. In this paper we obtained an estimation of a measure of algebraic independence of the values at algebraic points of the Jacobi elliptic function $\mathrm{sn}(z)$, which corresponds to an estimation in [16].

Theorem 3. *Let* $\mathrm{sn}(z)$ *be the Jacobi elliptic function,* $\varkappa$ *be an algebraic number* ($\varkappa \neq 0; 1$); $\alpha_1, \ldots, \alpha_n$ *be algebraic numbers linearly independent over* $\mathbb{Q}$; $\beta_1, \ldots, \beta_k$ *numbers such that* $\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n)$ *be algebraic over* $\mathbb{Q}(\beta_1, \ldots, \beta_k)$. *Then for any polynomial* $A \in \mathbb{Z}[x_1, \ldots, x_k]$, $A \not\equiv 0$, *whose degree does not exceed* D *and the absolute value of its coefficients does not exceed* H, *the inequality*

$$|A(\beta_1, \ldots, \beta_k)| \geqslant H^{-c_1 D^k} \tag{1}$$

holds, where H *and* D *are positive numbers such that*

$$\ln \ln H \geqslant c_2 D^k \ln(D+1), \quad D \geqslant 1.$$

We note that $c_1, c_2, \ldots$ are positive constants depending only on $\varkappa$ and $\alpha_1, \ldots, \alpha_k$.

Theorem 2 implies $k \geqslant [n/2]$. For $k = [n/2]$ an analogous result was proved in [6]. Similar estimations can be obtained for an other Jacobi elliptic functions.

In section 2 we include statements needed for the proof of Theorem 3. In section 3 Theorem 4 is proved, from which the proof of estimation (1) follows.

2. AUXILIARY STATEMENTS

The proof of Theorem 3 is based on Nesterenko's method explained in [9–17]. This method generalizes, on the base of algebraic technique, Gel'fond's method of the proof of algebraic independence.

Let us remind that the ideal I of the ring of polynomials is said to be unmixed, if all its primary components have identical dimension, equal to the dimension of an ideal I. Under the dimension $\dim I$ homogeneous ideal I we understand its projective dimension (see, for example, [27]). We use the notations $\deg I$ and $h(I)$ for degree and logarithmic height of the ideal I.

For any $\overline{\gamma} = (\gamma_1, \ldots, \gamma_m) \in \mathbb{C}^m$ we denote $|\overline{\gamma}| = \max_{1 \leqslant j \leqslant m} |\gamma_j|$.

Let $\mathbb{K} = \mathbb{Q}(\varkappa, \alpha_1, \ldots, \alpha_n)$, $\mathbb{K}[\overline{x}]$ be the ring of polynomials of variables $x_0, \ldots, x_m$ over $\mathbb{K}$. For any homogeneous unmixed ideal I, $I \subset \mathbb{K}[\overline{x}]$, and nonzero point $\overline{\gamma} \in \mathbb{C}^{m+1}$ one can define the magnitude of the ideal I at the point $\overline{\gamma}$, denoted $|I(\overline{\gamma})|$, analogous to the absolute value of $|P(\overline{\gamma})|$ for a polynomial P.

Lemma 1. *Let I be an unmixed, homogeneous ideal of the ring $\mathbb{K}[\overline{x}]$ with* $\dim I \geqslant 0$. *Suppose that $I = I_1 \cap \cdots \cap I_s$ is the reduced primary decomposition with $\mathfrak{p}_j = \sqrt{I_j}$ and k_j the exponent of I_j. Let $\overline{\gamma} \in \mathbb{C}^{m+1}$, $\overline{\gamma} \neq 0$. Then*

1) $\sum_{j=1}^{s} k_j \deg \mathfrak{p}_j = \deg I;$

2) $\sum_{j=1}^{s} k_j h(\mathfrak{p}_j) \leqslant h(I) + m^2 \deg I;$

3) $\sum_{j=1}^{s} k_j \ln |\mathfrak{p}_j(\overline{\gamma})| \leqslant \ln |I(\overline{\gamma})| + m^3 \deg I.$

The proof of Lemma 1 is similar to the proof of Proposition 1.2 in [17].

For any polynomial P with integer coefficients a_i we use the notation $\deg P$ for the degree of the polynomial P, $|P| = \max |a_i|$, $h(P)$ the logarithmic height of the polynomial P, $|P|_{\overline{\gamma}} = |P(\overline{\gamma})| \cdot |P|^{-1} \cdot |\overline{\gamma}|^{-\deg P}$ (see, for example, [17]).

For any two nonzero elements $\overline{\gamma} \in \mathbb{C}^{m+1}$ and $\overline{\vartheta} \in \mathbb{C}^{m+1}$, we denote the projective distance between these points by

$$|\overline{\gamma} - \overline{\vartheta}| = \left(\max_{0 \leqslant i < j \leqslant m} |\overline{\gamma}_i \overline{\vartheta}_j - \overline{\gamma}_j \overline{\vartheta}_i| \right) |\overline{\gamma}|^{-1} |\overline{\vartheta}|^{-1}.$$

Lemma 2. *Let $\mathfrak{p} \subset \mathbb{K}[\overline{x}]$ be a homogeneous prime ideal with* $\dim \mathfrak{p} \geqslant 0$. *Suppose that $Q \in \mathbb{K}[\overline{x}]$ is a homogeneous polynomial with $Q \notin \mathfrak{p}$. If $r = 1 + \dim \mathfrak{p} \geqslant 2$, then there exists a homogeneous unmixed ideal $J \subset \mathbb{K}[\overline{x}]$ such that its zeros coincide with those of the ideal* $(\mathfrak{p}, Q)$, $\dim J = \dim \mathfrak{p} - 1$ *and*

1) $\deg J \leqslant \deg \mathfrak{p} \cdot \deg Q;$

2) $h(J) \leqslant h(\mathfrak{p}) \deg Q + h(Q) \deg \mathfrak{p} + m(r+1) \deg \mathfrak{p} \cdot \deg Q;$

3) for any point $\overline{\gamma} \in \mathbb{C}^{m+1}$ and $\rho = \min |\overline{\gamma} - \overline{\vartheta}|$, where the minimum is taken over on all nontrivial zeros $\overline{\vartheta} \in \mathbb{C}^{m+1}$ of the ideal $\mathfrak{p}$, we have

$$\ln |J(\overline{\gamma})| \leqslant \ln \delta + h(Q) \deg \mathfrak{p} + h(\mathfrak{p}) \deg Q + 11m^2 \deg \mathfrak{p} \cdot \deg Q, \qquad (2)$$

where

$$\delta = \begin{cases} |Q|_{\overline{\gamma}}, & \text{if } \rho < |Q|_{\overline{\gamma}}, \\ |\mathfrak{p}(\overline{\gamma})|, & \text{if } \rho \geqslant |Q|_{\overline{\gamma}}. \end{cases}$$

Inequality (2) *holds for* $r = 1$, *if in this case we formally assume* $|J(\overline{\gamma})| = 1$.

The proof of Lemma 2 is similar to the proof of Proposition 1.4 [17].

Connection between the magnitude $|I(\overline{\gamma})|$ and the distance the point $\overline{\gamma}$ to the variety of zeros of a homogeneous unmixed ideal I is indicated in the following statement.

Lemma 3. *Let* $I \subset \mathbb{K}[\overline{x}]$ *be a homogeneous unmixed ideal,* $r = 1 + \dim I$, $r \geqslant 1$. *Then for any nonzero point* $\overline{\gamma} \in \mathbb{C}^{m+1}$ *there exists a zero* $\overline{\vartheta} \in \mathbb{C}^{m+1}$ *of the ideal* I *such that*

$$\deg I \cdot \ln\, |\overline{\gamma} - \overline{\vartheta}\,| \leqslant \frac{1}{r}\,(\ln |I(\overline{\gamma})| + h(I)) + 4m^3 \deg I.$$

The proof of Lemma 3 is similar to that of Proposition 1.5 [17].

Lemma 4. *Let* Q *be a homogeneous polynomial of the ring* $\mathbb{K}[\overline{x}]$ *and* $\overline{\gamma}$, $\overline{\vartheta} \in \mathbb{C}^{m+1}$ *are nonzero points and* $Q(\overline{\vartheta}) = 0$. *Then*

$$|Q\,|_{\overline{\gamma}} \leqslant\, |\overline{\gamma} - \overline{\vartheta}\,| \cdot e^{(2m-1)\deg Q}.$$

The proof of Lemma 4 is similar to the proof of Corollary 1 from Lemma 1.11 [17].

The definitions of the above introduced notions as well as complete formulations, and the proofs of necessary properties of the ideals can be found in [17]. The definition of the basic concepts related to the measure of algebraic independence, and also many auxiliary statements can be found, for example, in [3, 18, 22].

Further we shall adhere to standard notations in the theory of Weierstrass and Jacobi functions (see, for example, [4, 24]). Let us remind some properties of the Jacobi elliptic functions. Let $\mathrm{sn}(z)$ be the elliptic Jacobi function, its elliptic module $\varkappa$ be an algebraic number, $\varkappa \neq 0; 1$. We denote by $4K$ and $2iK'$ its fundamental periods, $\alpha_1, \ldots, \alpha_n$ algebraic numbers linearly independent over $\mathbb{Q}$. We denote

$$\omega_0 = 1,\ \omega_{2j-1} = \mathrm{sn}(\alpha_j),\ \omega_{2j} = \mathrm{sn}'(\alpha_j),\ 1 \leqslant j \leqslant n;\ m = 2n.$$

Lemma 5. *Let* $\overline{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, $\overline{l} \neq 0$. *Then there exists homogeneous polynomials* $S_{\overline{l}}$, $T_{\overline{l}}$, $U_{\overline{l}} \in \mathbb{Z}[\varkappa^2][\overline{x}]$ *such that the following conditions are satisfied:*

1) the total degree is at most $c_3(l_1^2 + \cdots + l_n^2)$;

2) the size is at most $\exp(c_3(l_1^2 + \cdots + l_n^2))$

and

$$\mathrm{sn}(l_1\alpha_1 + ... + l_n\alpha_n) = T_{\bar{l}}(\overline{\omega})/U_{\bar{l}}(\overline{\omega}),$$

$$\mathrm{sn}'(l_1\alpha_1 + ... + l_n\alpha_n) = S_{\bar{l}}(\overline{\omega})/U_{\bar{l}}(\overline{\omega}).$$

In addition, the inequality

$$|U_{\bar{l}}(\overline{\omega})| \geqslant \exp(-c_3(l_1^2 + \cdots + l_n^2))$$

holds.

The proof of Lemma 5 is similar to the proof of Lemma 7.2 [1] and Lemma 1 [8].

Let Λ be a two-dimensional lattice in $\mathbb{C}$ and $\mathfrak{M} \subset \mathbb{C}$. A number α is said to be $\mathfrak{M}$-admissible for Λ, if α is congruent modulo Λ to some point of the set $\mathfrak{M}$.

Lemma 6. *Let $\alpha_1, \ldots, \alpha_n$ be complex numbers linearly independent over $\mathbb{Q}$ modulo the lattice Λ. In the complex plane there is a compact set $\mathfrak{M}$, not containing points of Λ, with the property that for any real numbers L_1, $L_1 \geqslant L_0$, among the points $l_1\alpha_1 + \cdots + l_n\alpha_n$, $0 \leqslant l_j < L_1$, $l_j \in \mathbb{Z}$, there are not less than $\frac{7}{8}L_1^n$ of points being $\mathfrak{M}$-admissible. Here, the set $\mathfrak{M}$ and the boundary of L_0 depends only on Λ and $\alpha_1, \ldots, \alpha_n$.*

The proof of Lemma 6 is similar to that of Lemma 5 [20].

We shall apply Lemma 6, assuming that the numbers $\alpha_1, \ldots, \alpha_n$ satisfy the conditions of Theorem 3 and Λ is the lattice of half-periods of $\mathrm{sn}(z)$, and further the set $\mathfrak{M}$ is defined for these numbers and lattices.

Lemma 7. *The functions*

$$\sigma((z + iK')/\sqrt{e_1 - e_3}), \quad \sigma((z + iK')/\sqrt{e_1 - e_3})\,\mathrm{sn}(z)$$

are entire and for an arbitrary $M_0 > 1$ we have

$$|\sigma((z + iK')/\sqrt{e_1 - e_3})\,\mathrm{sn}(z)|_{|z| \leqslant M_0} \leqslant c_4^{M_0^2},$$

$$|\sigma((z + iK')/\sqrt{e_1 - e_3})|_{|z| \leqslant M_0} \leqslant c_4^{M_0^2}.$$

If δ is the distance from z_0 to the nearest pole of $\mathrm{sn}(z)$ *and $|z_0| \leqslant M_0$, then $|\sigma((z + iK')/\sqrt{e_1 - e_3})| \geqslant \delta c_5^{-M_0^2}$.*

The proof of Lemma 7 is similar to that of Lemma 7.1 [7].

Lemma 8. *Let R_1, $R_2 \in \mathbb{R}$, $8 < 4R_1 < R_2$, $f(z)$ be a function analytic in the disk $|z| \leqslant R_2$; Let E be a set from N^2 of points belonging to the disk $|z| \leqslant R_1$ and distance between any pair points is at least ε, $0 < \varepsilon < 1$. Then*

$$|f(z)|_{|z|\leqslant R_1} \leqslant 2|f(z)|_{|z|\leqslant R_2} \left(\frac{4R_1}{R_2}\right)^{N^2 S} + \\ + 2NR_1^{-1}\left(\frac{33R_1}{\epsilon N}\right)^{N^2 S} \max_{x\in E,\, 0\leqslant s\leqslant S} \left|\frac{f^{(s)}(x)}{s!}\right|.$$

The proof of Lemma 8 can be found in [21].

3. PROOF OF THE THEOREM 3

In this section Theorem 3 will be deduced from the following statement.

Theorem 4. *For each integer r, $1 \leqslant r \leqslant k$, there exist constants $\mu_r \geqslant 0, \gamma_r \geqslant 1$ such that for any numbers D and H satisfying the inequality*

$$\ln\ln H \geqslant \gamma_r D^k \ln(D+1), \quad D \geqslant 1,$$

and for any homogeneous unmixed ideal $I \subset \mathbb{K}[\overline{x}]$ with the conditions $\dim I = r-1$, $\deg I \leqslant D^{1-r+k}$, $h(I) \leqslant D^{-r+k}\ln H$ *the inequality*

$$\ln|I(\overline{\omega})| \geqslant -\mu_r(Dh(I) + \deg I \ln H)D^{r-1}$$

holds.

We shall show how Theorem 4 can be derived from Theorem 3.

For any polynomial $B \in \mathbb{Z}[x_1, x_3, \ldots, x_{2n-1}]$ such that

$$B(\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n)) \neq 0,$$

we denote by

$$C(x_0, x_1, x_3, \ldots, x_{2n-1}) = x_0^{\deg B} B\left(\frac{x_1}{x_0}, \frac{x_3}{x_0}, \ldots, \frac{x_{2n-1}}{x_0}\right).$$

Then

$$C(1, \mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n)) = B(\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n)). \tag{3}$$

Let $\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_k)$ be algebraically independent. Then there are polynomials R_{k+i}, $Q_j \in \mathbb{K}[x_1, \ldots, x_m]$, $i = 1, \ldots, n-k$, $j = 1, \ldots, n$, such that $R_{k+i}(\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_k), \mathrm{sn}(\alpha_{k+i})) = 0$ and $Q_j(\mathrm{sn}(\alpha_j), \mathrm{sn}'(\alpha_j)) = 0$. The ideal $\mathfrak{J}$, generated by homogeneous polynomials corresponding to R_{k+i} and Q_j, has dimension k. Let $\mathfrak{p}$ be a homogeneous prime ideal generated by all homogeneous polynomials in the ring $\mathbb{K}[\overline{x}]$ that vanish at $\overline{\omega}$. Then $\mathfrak{J} \subset \mathfrak{p}$ and from the assumption of Theorem 4 it follows that $\dim \mathfrak{p} = k$.

We denote by J the homogeneous unmixed ideal in the ring $\mathbb{K}[\overline{x}]$, which is constructed for $\mathfrak{p}$ and C in accordance with Lemma 2. Then $\dim J = \dim \mathfrak{p} - 1 = k - 1$, $\deg J \leqslant c_6 \deg C$, $h(J) \leqslant c_7\big(h(C) + \deg C\big)$. In addition,

$$\ln |J(\overline{\omega})| \leqslant \ln \|C\|_{\overline{\omega}} + c_8\big(h(C) + \deg C\big). \tag{4}$$

Applying Theorem 4 to the ideal J with $r = k$ and taking into account (3), from (4) we obtain the inequality

$$\ln |B(\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n))| \geqslant -c_9(Dh(J) + \deg J \ln H)D^{k-1}. \tag{5}$$

It follows from the assumptions of Theorem 3 that all numbers $\beta_1, \ldots, \beta_k$ are algebraic over the field $\mathbb{Q}(\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n))$. Then for some integer algebraic over the ring $\mathbb{Q}[\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n)]$ number d all numbers $d\beta_i$ are algebraic integers over $\mathbb{Q}[\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n)]$. Define

$$B(\mathrm{sn}(\alpha_1), \ldots, \mathrm{sn}(\alpha_n)) = \mathrm{Norm}(d^{\deg A} A(\beta_1, \ldots, \beta_k)). \tag{6}$$

Since $\deg B \leqslant c_{10} \deg A$, $\ln H(B) \leqslant c_{11} \ln H(A)$, we obtain the estimation of Theorem 3 taking into account (5) and (6).

We shall prove Theorem 4 by induction on r. Let r_0 , $1 \leqslant r_0 \leqslant k$, be the least integer for which the assertion of Theorem 4 is no longer true. We choose and fix a sufficiently large integer number λ.

Lemma 9. *The set of numbers D, for which there exists a prime homogeneous ideal $\mathfrak{p}$ in ring $\mathbb{K}[\overline{x}]$ with*

$$\ln \ln H \geqslant \gamma_{r_0} D^k \ln(D+1), \quad D \geqslant 1,$$
$$\mathfrak{p} \cap \mathbb{Z} = (0), \quad \dim \mathfrak{p} = r_0 - 1, \quad \deg \mathfrak{p} \leqslant D^{1-r_0+k}, \quad h(\mathfrak{p}) \leqslant 2D^{-r_0+k} \ln H,$$
$$\ln |\mathfrak{p}(\overline{\omega})| < -\lambda^{8k^2+8k+4}(Dh(\mathfrak{p}) + \deg \mathfrak{p} \ln H)D^{r_0-1}$$

is unbounded.

Indeed, otherwise, the inequality

$$\ln |\mathfrak{p}(\overline{\omega})| \geqslant -c_{12}(Dh(\mathfrak{p}) + \deg \mathfrak{p} \ln H)D^{r_0-1} \tag{7}$$

would hold with a certain positive constant c_{12} for all homogeneous prime ideals $\mathfrak{p} \subset \mathbb{K}[\overline{x}]$, $\dim \mathfrak{p} = r_0 - 1$.

Applying now Lemma 1 to an arbitrary homogeneous unmixed ideal $I \subset \mathbb{K}[\overline{x}]$ of dimension $r_0 - 1$, taking into account (7) we obtain

$$\ln|I(\overline{\omega})| \geqslant -c_{13}(Dh(I) + \deg I \ln H)D^{r_0 - 1}.$$

But it contradicts the assumption that the assertion of Theorem 4 does not hold for the ideals of dimension $r-1$ and proves Lemma 9.

Let now H be sufficiently large, and let a number D and prime homogeneous ideal $\mathfrak{p}$ of dimension $r-1$ satisfy the conditions of Lemma 9.Let us define the number M by the relation

$$\lambda D^k M \ln M = \min(\lambda^{8k^2+8k+4}(Dh(\mathfrak{p}) + \deg\mathfrak{p} \ln H)D^{r-1}, \ \frac{1}{2}\ln(1/\rho)), \quad (8)$$

where ρ is defined in Lemma 2. Using Lemma 3 and (8), we have

$$M \ln M \geqslant \ln H(\ln\ln H)^{-1}. \quad (9)$$

From Lemma 4 and (8) we obtain:

Lemma 10. *Let a homogeneous polynomial $P \in \mathbb{K}[\overline{x}]$ be contained in the ideal $\mathfrak{p}$ and satisfies the inequality*

$$h(P) + (2m+1)\deg P \leqslant \lambda D^k M \ln M.$$

Then

$$|P(\overline{\omega})||\overline{\omega}|^{-\deg P} \leqslant \exp\left(-\lambda D^k M \ln M\right).$$

From Lemma 5, Lemma 10 and (9) we obtain:

Lemma 11. *If $\overline{l} \in \mathbb{Z}^n \backslash \{0\}$, $|l_j| \leqslant \lambda^{4k+1}D^{1/2}$, then $U_{\overline{l}}(\overline{x}) \notin \mathfrak{p}$.*

Using Lemma 2, Lemma 9, (8), (9) and the inductive assumption, we obtain the following estimation:

Lemma 12. *Let a homogeneous polynomial $P \in \mathbb{K}[\overline{x}]$ be not contained in the ideal $\mathfrak{p}$ and also satisfy the inequality*

$$\deg P \leqslant \lambda^{8k+5}D, \quad h(P) \leqslant \lambda^{8k+5}\ln H.$$

Then

$$|P(\overline{\omega})||\overline{\omega}|^{-\deg P} \geqslant \exp\left(-\frac{1}{4}\lambda D^k M \ln M\right).$$

Having applied Lemma 12 to each basis ideal, we obtain the following statement.

Lemma 13. *Let $\mathfrak{I}$ be the ideal in $\mathbb{K}[\overline{x}]$ generated by all homogeneous polynomials $P \in \mathbb{K}[\overline{x}]$ such that $P(\overline{\omega}) = 0$. Then $\mathfrak{I} \subset \mathfrak{p}$.*

We denote

$$L = \left[\lambda^{4k} D^{1/2}\right],\ K_0 = \left[D^k M\right],\ K_1 = \lambda^2,\ S = \left[\lambda^{1-8k^2} M\right]. \qquad (10)$$

Lemma 14. *There exist homogeneous polynomials $A_{\overline{k}} \in \mathbb{K}[\overline{x}]$, $\overline{k} = (k_0, k_1)$, $0 \leqslant k_0 < K_0$, $0 \leqslant k_1 < K_1$, with*

1) $\deg A_{\overline{k}} \leqslant \lambda^{8k+4} D$, $\ln H(A_{\overline{k}}) \leqslant 4\lambda^{1-8k^2} M \ln M$;

2) at least one of these polynomials is not contained in $\mathfrak{p}$;

3) if

$$F(z) = \sum_{k_0=0}^{K_0-1} \sum_{k_1=0}^{K_1-1} A_{\overline{k}}(\overline{\omega}) z^{k_0} \operatorname{sn}^{k_1}(z),$$

then for all numbers $s, 0 \leqslant s \leqslant S$, and all $\mathfrak{M}$-admissible points $l_1\alpha_1 + \cdots + l_n\alpha_n$, $l_j \in \mathbb{Z}$, $0 \leqslant l_j < L$, the inequality

$$|F^{(s)}(l_1\alpha_1 + \cdots + l_n\alpha_n)| \leqslant \exp\left(-\frac{1}{2}\lambda D^k M \ln M\right) \qquad (11)$$

holds.

The proof of Lemma 14 is similar to that of Lemma 10 in [16]. By analogy with [1] let us define two polynomials $P_1 \in \mathbb{Z}[\varkappa^2][t_1, t_2, t_3, t_4]$, $P_2 \in \mathbb{Z}[\varkappa^2][t_1, t_2]$ so that

$$\operatorname{sn}(z + w) = P_1(\operatorname{sn} z, \operatorname{sn}' z, \operatorname{sn} w, \operatorname{sn}' w) P_2^{-1}(\operatorname{sn} z, \operatorname{sn} w)$$

and $\deg P_1 = 2$, $\deg P_2 = 4$. Then there exist polynomials $G_{t,k,l} \in \mathbb{Z}[\varkappa^2][t_1, t_2]$ such that $\deg_{t_2} G_{t,k,l}[t_1, t_2] \leqslant 1$ and

$$G_{t,k,l}(t_1, t_2) \equiv \left(\frac{d}{dw}\right)^s (P_1^k(t_1, t_2, \operatorname{sn} w, \operatorname{sn}' w) P_2^l(t_1, \operatorname{sn} w))_{w=0},$$

where $\equiv$ stands for the congruence modulo $t_2^2-(1-t_1^2)(1-\varkappa^2t_1^2)$ in the ring $\mathbb{Z}[\varkappa^2][t_1,t_2]$. We obtain

$$F^{(s)}(z)=\sum_{t=0}^{s}\binom{s}{t}\left(\frac{d}{dw}\right)^{s-t}\left[P_2^{-K_1}(\mathrm{sn}(z),\varphi(w))\right]_{w=0}\times$$

$$\times\left\{\sum_{k_0=0}^{K_0-1}\sum_{k_1=0}^{K_1-1}A_{\overline{k}}(\overline{\omega})\sum_{i=0}^{t}\left[\binom{k_0}{i}\frac{t!}{(t-i)!}z^{k_0-i}G_{t-i,k_1,K_1-k_1}(\mathrm{sn}(z),\mathrm{sn}'(z))\right]\right\}.$$

For all integer numbers $s,l_1,\dots l_n$, $0\leqslant s<S$, $0\leqslant l_j<L$, we denote

$$R_{s,\overline{l}}(\widetilde{x})=\sum_{k_0=0}^{K_0-1}\sum_{k_1=0}^{K_1-1}B_{\overline{k}}(\widetilde{x})\sum_{i=0}^{t}\left[\binom{k_0}{i}\frac{t!}{(t-i)!}(l_1\alpha_1+\dots+l_n\alpha_n)^{k_0-i}\times\right.$$

$$\left.\times\, U_{\overline{l}}^{10K_1}(1,\widetilde{x})G_{t-i,k_1,K_1-k_1}(T_{\overline{l}}(1,\widetilde{x})/U_{\overline{l}}(1,\widetilde{x}),S_{\overline{l}}(1,\widetilde{x})/U_{\overline{l}}(1,\widetilde{x}))\right],$$

where $\widetilde{x}=(x_1,\dots x_m)$, polynomials $G_{t,k,l}(x,y)$ are defined for $\mathrm{sn}(z)$ in the same way as in [16]. Applying to the system

$$R_{s,\overline{l}}(\widetilde{x})=0,\ 0\leqslant s<S,\ 0\leqslant l_j<L,\ j=1,\dots,n, \tag{12}$$

Ziegel's Lemma (see, for example, [22]), we obtain exist that there polynomials, not of all equal to zero, $B_{\overline{k}}(\widetilde{x})\in\mathbb{K}[\widetilde{x}]$ such, that

$$\deg B_{\overline{k}}\leqslant\lambda^{8k+4}D,\ \ln H(B_{\overline{k}})\leqslant 3\lambda^{1-8k^2}M\ln M.$$

Like in [16], let ${}^a\mathfrak{p}$ be the dehomogenization of the ideal $\mathfrak{p}$. We denote by the letter u the least integer number such that

1) there exists a vector

$$\overline{u}=(u_1,\dots,u_m)\in\mathbb{Z}^m,u_i\geqslant 0,u_1+\dots+u_m=u;$$

2) there exists indices k_0^*,k_1^* such that $\delta_{\overline{u}}B_{k_0^*,k_1^*}\notin{}^a\mathfrak{p}$, where

$$\delta_{\overline{u}}=\prod_{i=1}^{m}\frac{1}{u_i!}\left(\frac{\partial}{\partial x_i}\right)^{u_i}.$$

Then from (12) it follows that $\partial_{\overline{u}}R_{s,\overline{l}}(\widetilde{x})\in{}^a\mathfrak{p}$.

Let $A_{\overline{k}}(\overline{x})$ be the homogenization of $\partial_{\overline{u}}B_{k_0,k_1}(\widetilde{x})$. We denote

$$Q_{s,\overline{l}}(\overline{x})=\sum_{k_0=0}^{K_0-1}\sum_{k_1=0}^{K_1-1}A_{\overline{k}}(\overline{x})\sum_{i=0}^{t}\left[\binom{k_0}{i}\frac{t!}{(t-i)!}(l_1\alpha_1+\dots+l_n\alpha_n)^{k_0-i}\times\right.$$

$$\times U_{\bar{l}}^{10K_1}(\overline{x})G_{t-i,k_1,K_1-k_1}(T_{\bar{l}}(\overline{x})/U_{\bar{l}}(\overline{x}), S_{\bar{l}}(\overline{x})/U_{\bar{l}}(\overline{x}))\Big].$$

As $\partial_{\overline{u}}R_{s,\bar{l}}(\widetilde{x}) \in {}^a\mathfrak{p}$, then $Q_{s,\bar{l}}(\overline{x}) \in \mathfrak{p}$ and, using Lemma 10, we obtain the estimation

$$|Q_{s,\bar{l}}(\overline{\omega})| < \exp\left(-\frac{2}{3}\lambda D^k M \ln M\right).$$

From this estimation similar by to [16] the estimation of Lemma 14follows.

We define

$$G(z) = \sigma((z + iK')/\sqrt{e_1 - e_3})F(z).$$

From properties of $\mathrm{sn}(z)$, using Lemma 7, Lemma 8, (10) and (11), we obtain the following statement.

Lemma 15. *In the disk $|z| \leqslant \lambda^{4k+2}D^{1/2}$ the inequality*

$$|G(z)| \leqslant \exp\left(-\frac{3}{8}\lambda D^k M \ln M\right)$$

holds.

Using Lemmas 5, 6, 7, 9, 12, 13, 15 we obtain:

Lemma 16. *For all numbers s, $l_1, \ldots, l_n$ such that $0 \leqslant s \leqslant S$, $0 \leqslant l_j \leqslant \lambda^{4k+1}D^{1/2}$ and the point $l_1\alpha_1 + \cdots + l_n\alpha_n$ is $\mathfrak{M}$-admissible, we have*

$$|F^{(s)}(l_1\alpha_1 + \cdots + l_n\alpha_n)| \leqslant \exp\left(-\frac{1}{3}\lambda D^k M \ln M\right).$$

Let $\mathcal{R}$ be a quotient-ring $\mathbb{K}[\widetilde{x}]/{}^a\mathfrak{p}$, η_i be the images of x_i, $1 \leqslant i \leqslant m$, $\eta = (\eta_1, \ldots, \eta_m)$, $\mathcal{L}$ be the field of quotients $\mathcal{R}$. It follows from Lemma 11 that $U_{\bar{l}}(\overline{\eta}) \notin \mathfrak{p}$. We define $\overline{\xi}_{\bar{l}} = (\overline{\xi}_{\bar{l},1}, \overline{\xi}_{\bar{l},2}, \overline{\xi}_{\bar{l},3})$ as follows:

$$\overline{\xi}_{\bar{l}} = (l_1\alpha_1 + \cdots + l_n\alpha_n,\ T_{\bar{l}}(\overline{\eta})/U_{\bar{l}}(\overline{\eta}),\ S_{\bar{l}}(\overline{\eta})/U_{\bar{l}}(\overline{\eta})), \quad \overline{\xi}_{\bar{l}} \in \mathcal{L}^3.$$

As the point $(\mathrm{sn}(l_1\alpha_1+\cdots+l_n\alpha_n), \mathrm{sn}'(l_1\alpha_1+\cdots+l_n\alpha_n)), \bar{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, $\bar{l} \neq 0$, lays on the curve $y^2 = (1 - x^2)(1 - \varkappa^2 x^2)$, the equality

$$U_{\bar{l}}^2(\overline{\omega})S_{\bar{l}}^2(\overline{\omega}) = (U_{\bar{l}}^2(\overline{\omega}) - T_{\bar{l}}^2(\overline{\omega}))(U_{\bar{l}}^2(\overline{\omega}) - \varkappa^2 T_{\bar{l}}^2(\overline{\omega}))$$

holds. From here and from Lemma 13 it follows that the two last coordinates of the points $\overline{\xi}_{\bar{l}}$ satisfy the equation $\xi_{\bar{l},3}^2 = (1 - \xi_{\bar{l},2}^2)(1 - \varkappa^2 \xi_{\bar{l},2}^2)$.

Lemma 17. *Let $\overline{\xi}_i = (\xi_{i1}, \xi_{i2}, \xi_{i3})$, $1 \leqslant i \leqslant N_2$, be points with distinct first coordinates, $R \in \mathcal{L}[z,x]$, $\deg_z R \leqslant L_2$, $\deg_x R \leqslant L_3$. Then*

$$\sum_{i=1}^{N_2} \operatorname{ord}_{\overline{\xi}_i} R \leqslant (4L_3+2)L_2 + 2N_2L_3.$$

The proof of Lemma 17 is similar to that of Lemma 16 [16].

Let

$$\mathcal{D} = \frac{\partial}{\partial z} + y\frac{\partial}{\partial x} + (2\varkappa^2x^3 - (1+\varkappa^2)x)\frac{\partial}{\partial y}$$

be the differential operator in the ring $\mathcal{L}[z,x,y]$. Using Lemmas 5 and 16, we obtain:

Lemma 18. *For each $s, l_1, \dots, l_n$, $0 \leqslant s \leqslant S, 0 \leqslant l_j \leqslant \lambda^{2n+1}D^{1/2}$, such that the point $l_1\alpha_1 + \dots + l_n\alpha_n$ is $\mathfrak{M}$-admissible, the equality*

$$\mathcal{D}^s F(z)|_{l_1\alpha_1+\dots+l_n\alpha_n} = 0$$

holds.

In order to complete the proof of Theorem 4 is suffices to reduce an inconsistency the estimations in Lemmas 17and 18. For the polynomial F constructed in Lemma 14, according to Lemma 18we have

$$\sum \operatorname{ord}_{\overline{\xi}_i} F \geqslant \frac{1}{2}\lambda^{2k+1}D^kM.$$

According to Lemma 17 this sum does not exceed the magnitude $6\lambda^2D^{n/2}M$. The obtained contradiction for $n > 1$ completes the proof of Theorem 4.

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ПРО МІРУ АЛГЕБРИЧНОЇ НЕЗАЛЕЖНОСТІ ЗНАЧЕНЬ ЕЛІПТИЧНИХ ФУНКЦІЙ ЯКОБІ. I

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У роботі отримано оцінку міри алгебричної незалежності значень у різних алгебричних точках еліптичної функції Якобі $\mathrm{sn}(z)$.