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ON OPTIMAL BOUNDARY CONTROL OF NON-HOMOGENEOUS STRING VIBRATIONS UNDER IMPULSIVE CONCENTRATED PERTURBATIONS WITH DELAY IN CONTROLS

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A. Khurshudyan, *On Optimal Boundary Control of Non-Homogeneous String Vibrations under Impulsive Concentrated Perturbations with Delay in Controls*, Math. Bull. Shevchenko Sci. Soc. **10** (2013), 203–209.

An optimal boundary control problem in a finite time-interval is considered for a non-homogeneous string, that is fixed at the left end-point. The string vibrates under external impulsive impacts concentrated at isolated points t and acting in discrete moments of the considered time-interval. The control process is carried out by the Dirichlet boundary controls containing a constant delay, at that a functional describing the linear momentum of controls in considered time-interval is taken as the control process optimality criterion. By means of the Fourier real generalized integral transform, solving of the problem is reduced to an interpolation problem with respect to an unknown Fourier transform of function, which in its turn is reduced to corresponding system of trigonometric moments problem with respect to an unknown function. The optimal resolving controls are obtained explicitly by means of generalized functions and it is shown that the influence of external impulsive impacts can be best removed by boundary impulsive controls. Conditions of the string controllability in the space of Lebesgue measurable functions are obtained.

А. Хуршудян. *Про оптимальне граничне керування неоднорідною струною під дією імпульсних збурень із запізненням у керуванні* // Мат. вісник НТШ. — 2013. — Т.10. — С. 203–209.

На скінченному інтервалі часу розглядаємо задачу оптимального керування для неоднорідної струни, що закріплена в одному з кінців. Струна здійснює коливання під дією зовнішніх імпульсних ударів в деяких точках струни у фіксовані моменти часу на вказаному інтервалі. Коливання контролюються функцією в крайових умовах Діріхле, яка містить запізнення. Крім того, критерієм оптимальності є функціонал, який відповідає лінійному моменту модуля функції керування за часом. За допомогою дійсного перетворення Фур'є задачу редукують до інтерполяційної задачі для образу Фур'є невідомої функції, яку в свою чергу зводять до проблеми тригонометричних моментів. Оптимальне керування знайдено у явній формі в термінах узагальнених функцій.

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Introduction

Control problems for vibrating systems with variable distributed parameters are especially important in view of almost-periodicity of that vibrations. Although control problems for non-homogeneous strings were investigated very rarely (see e.g. [1]–[5] and the references cited therein). In [1] explicit form of Dirichlet boundary control functions are found, as well as necessary and sufficient conditions on initial and terminal data for boundary controllability of free vibrations of non-homogeneous string controlled on both ends are obtained. In [2, 3] boundary null– and approximate null-controllability problems in modified Sobolev spaces are considered for non–homogeneous string, when control processes are carried out by Neumann and Dirichlet boundary conditions respectively, at that in [2] the deflection of the string is fixed at the left end–point and in [3] the velocity of the string is fixed at the right end–point. In [4, 5] optimal control problems are considered for non-homogeneous string which is fixed at the left end–point and vibrating under impulsive concentrated perturbations, and non-homogeneous rod which is fixed at the left end–point and is free at the right end–point, respectively, while control process is carried out by mixed boundary conditions in [4] and by external control impacts, distributed on the rod arbitrarily, in [5]. The problem is reduced to a truncated system of moments problem and solved explicitly by means of generalized functions. Controllability conditions of the string and rod in the sense of the Lebesgue space L^1 are obtained for the initial and terminal data.

Here we concern with boundary optimal control problem for non-homogeneous wave equation with variable coefficients, describing forced vibrations of non–homogeneous string, that is fixed at the left end–point, under impulsive concentrated perturbations, at that control process is carried out by Dirichlet boundary controls containing constant delay at the right end–point of the string. A functional describing the summary "linear momentum" of controls in the considered time interval is taken as a control process optimality criterion.

Throughout this paper we shall call a real function *admissible control* if it satisfies necessary and sufficient conditions of controlled system needed solution existence. We shall say, that a controlled system is *fully controllable* in a given space of functions, if there exists an admissible control, which resolves posed optimal control problem for the system in terms of that space.

1. Statement of the problem

We deal with one-dimensional wave equation with $x-$dependent coefficients

$$\frac{\partial}{\partial x}\left[T_0(x)\frac{\partial w(x,t)}{\partial x}\right]-\rho(x)\frac{\partial^2 w(x,t)}{\partial t^2}=P(x,t),\ (x,t)\in(0,l)\times(t_0,T), \tag{1}$$

subjected to the following Dirichlet boundary conditions:

$$w(0,t)\equiv 0,\ w(l,t)=u(t-\tau);\ t\in[t_0,T], \tag{2}$$

at that $\rho(x)$ and $T_0(x)$ are reasonable functions, such that the usual existence theorems for physically acceptable solution (generalized solutions may also be acceptable) $w=w(x,t)$ hold, $\tau>0$ is constant delay, at that obviously $2\tau<T-t_0$. It is also supposed, that by boundary function $u(t)$ control process of the string under study is carried out.

In addition let us suppose, that external impacts $P(x,t)$ reads as follows:

$$P(x,t)=\sum_{j,s=1}^{m,n}P_{js}\delta\left(x-x_j\right)\delta(t-t_s),\ \ (x,t)\in(0,l)\times(t_0,T),$$

where $\delta(x)$ is Dirac's delta–function, $P_{js}=const, j=\overline{1;m},\ s=\overline{1;n},\ \{x_j\}_{j=1}^{m}\subset(0,l)$, $\{t_s\}_{s=1}^{n}\subset[t_0,T]$.

Note, that equation (1) particularly describes forced vibrations of a non-homogeneous string of density $\rho(x)$ and length l, caused by impulsive perturbations $P_{js},\ j=\overline{1;m},\ s=\overline{1;n}$, applied on isolated points $x_j\in(0,l)$ in discrete moments $t_s\in[t_0,T)$, at that $T_0(x)$ is the tensile force, and $w=w(x,t)$ is the deflection of the string from equilibrium state.

The following initial data are considered:

$$w(x,t_0)=w_{t_0}(x),\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=t_0}=\dot{w}_{t_0}(x);\ x\in[0,l]. \tag{3}$$

Our aim is to define optimal law of realization and find necessary and sufficient conditions for existence of boundary optimal control function $u^o(t)$ providing for solution of (1), satisfying boundary and initial conditions (2) and (3), the following end–point conditions:

$$w(x,T)=w_T(x),\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=T}=\dot{w}_T(x);\ x\in[0,l] \tag{4}$$

and minimizing control process optimality criterion

$$\kappa[u]=\int_{t_0}^{T-\tau}|u(t)|dt. \tag{5}$$

Note, that this functional evaluates summary "linear momentum" of boundary controls $u(t)$ in time–interval $[t_0,T-\tau]$ [6].

Without getting into details, note that all transmission conditions concerning boundary conditions (2) and initial and terminal data (3), (4), are supposed to be satisfied:

$$w_{t_0}(l)=0,\ \dot{w}_{t_0}(l)=0,\ w_T(l)=u(T-\tau),\ \dot{w}_T(l)=\left.\frac{du(t)}{dt}\right|_{t=T-\tau}.$$

2. Solution of the problem

In order to solve the problem under investigation, let us apply Fourier real generalized integral transform with respect to t variable to (1) and (2). Then, taking into account (3), (4) and properties of generalized Fourier integrals [7], after some algebraic transformations and simplifications, we shall respectively derive:

$$\frac{d}{dx}\left[T_0(x)\frac{d\overline{w}(x,\sigma)}{dx}\right]+\sigma^2\rho(x)\overline{w}(x,\sigma)=G(x,\sigma);\ (x,\sigma)\in(0,l)\times(-\infty,\infty), \tag{6}$$

$$\overline{w}(0,\sigma)\equiv0,\ \overline{w}(l,\sigma)=\overline{u}(\sigma)e^{i\sigma\tau}, \tag{7}$$

where

$$F_t[w]\equiv\overline{w}(x,\sigma)=\int_{-\infty}^{\infty}w_0(x,t)e^{i\sigma t}dt;\ w_0(x,t)=[H(t-t_0)-H(t-T)]w(x,t),$$

$$F_t[u] \equiv \overline{u}(\sigma) = \int_{-\infty}^{\infty} u_0(t)e^{i\sigma t}dt;\ u_0(t) = [H(t-t_0) - H(t+\tau-T)]u(t),$$

$F_t[\cdot]$ is the Fourier operator, σ is the spectral parameter of Fourier transform, $H(t)$ is the Heaviside unit step function,

$$G(x,\sigma) = G_1(x,\sigma) + i \cdot G_2(x,\sigma),\ G_p(x,\sigma) = \sum_{j,s=1}^{m,n} P_{js}\delta(x-x_j)\{\cos;\sin\}(\sigma t_s) + \mathrm{g}_p(x,\sigma),$$

$$\mathrm{g}_p(x,\sigma) = \rho(x)\Big[\dot{w}_T(x)\{\cos;\sin\}(\sigma T) - \dot{w}_{t_0}(x)\{\cos;\sin\}(\sigma t_0) + \\ + \sigma\big(w_T(x)\{\sin;-\cos\}(\sigma T) + w_{t_0}(x)\{-\sin;\cos\}(\sigma t_0)\big)\Big];\ p \in \{1,2\}.$$

We additionally suppose, that functions $T_0(x)$ and $\rho(x)$ satisfy conditions of existence for non-trivial oscillating fundamental solution of (6) [8]–[9]. Then, solution of control problem under study gives the following

Theorem 2.1. *Denote by U the set of admissible controls, for which $w(x,t)$ function is compactly supported in $[t_0,T]$. If $u \in U$, then it is defined as Fourier inverse transform of a function $\overline{u}(z)$, determined from system*

$$\overline{u}(z_k) = \Omega(l,z_k)e^{-iz_k\tau},\ k = 1,2,3,... \tag{8}$$

where points z_k are determined from characteristic equations

$$\lambda(l,z_k) - \lambda(0,z_k) - (\mu(l,z) - \mu(0,z_k)) = 2\pi k,\ k = 1,2,3,.... \tag{9}$$

Here

$$\Omega(x,\sigma) = -i\int_0^x K(x,\xi,\sigma)G(\xi,\sigma)d\xi,\ K(x,\xi,\sigma) = \frac{e^{i\left(\lambda(x,\sigma)-\lambda(\xi,\sigma)\right)} - e^{i\left(\mu(x,\sigma)-\mu(\xi,\sigma)\right)}}{\mu'(\xi,\sigma) - \lambda'(\xi,\sigma)},$$

and $\lambda = \lambda(x,\sigma)$ and $\mu = \mu(x,\sigma)$ are non–trivial real functions determining from Riccati equation [4],[5],[9] $\nu^2 + T_0(x)\nu' + \sigma^2\rho(x)T_0(x) = 0$, by $\nu(x,\sigma) = i \cdot T_0(x)\frac{d}{dx}\{\lambda;\mu\}$, and defining oscillating solution of (6), s.t. $\lambda'(x,\sigma) \neq \mu'(x,\sigma)$ for all $x \in [0,l]$ and $\sigma \in (-\infty,\infty)$.

Proof. One can easily see, that the set U should consist of $u(t)$ functions, which acts until some $t_* \in (t_0, T-\tau]$, i.e. are compactly supported, at least, in $[t_0, T-\tau]$. From the other hand, as the space of Lebesgue measurable functions $\mathrm{L}^1[t_0,T-\tau]$ is a Banach space with respect to norm (5), it is advisable to take as set of admissible controls the set of compactly supported in $[t_0,T-\tau]$ functions $U \subset \mathrm{L}^1[t_0,T-\tau]$.

It is well known [7], that if function $w(x,t)$ is compactly supported in $[-\theta,\theta]$ (every non–symmetric interval $[t_0,T]$ can be transformed into a symmetric one in the issue of a linear change of variable), then its Fourier generalized transform is an analytical entire function of complex variable $z = \sigma + i\gamma$, satisfying inequality

$$|z^\rho \cdot \overline{w}(x,z)| \leq A_\rho e^{\theta|\gamma|} \tag{10}$$

for all $x \in [0,1]$ and $\rho \geq 0$ with $A_\rho \geq 0$. On the other hand, as $P(x,t) \in \mathrm{L}^1[0,l] \times [t_0, T]$, the function $G(x,z)$ is also an analytical entire function of z, satisfying inequality (10) [7]. Thus, extending the oscillating solution of system (6)–(7) for all z, one can obtain that for fulfillment of aforesaid theorem conditions equalities (8)–(9) are necessary and sufficient. □

Note, that system of equalities (8) can be treated as interpolation conditions given in nodes z_k for determining the function $\overline{u}(z)$. Then, solution of interpolation problem (8) can be achieved by different efficient methods of interpolation, application of Fourier inverse transform to which will provide us required solution. However, here we will proceed in a different way– by separating both sides of relations (3.3) to real and imaginary parts. As a result of decomposition we will derive the following countable system of real equalities for determining desired function $u(t)$:

$$\int_{t_0}^{T-\tau} u(t)e^{-\gamma_k t}\cos(\sigma_k t)dt = M_{1k}, \int_{t_0}^{T-\tau} u(t)e^{-\gamma_k t}\sin(\sigma_k t)dt = M_{2k}, \quad k \in \mathbb{N}, \tag{11}$$

where

$$M_{1k} + iM_{2k} \equiv M_k = \Omega(l, \sigma_k + i\gamma_k)e^{(\gamma_k - i\sigma_k)\tau}.$$

In order to solve control problem under study one may use only truncated version of system (11) for some natural N. The convergence of truncated system‘s solution to solution of infinite system (11) is investigated as in [10].

Remark 2.2. Using the symmetry property of obtained Riccati equation: $\nu(x,-z) = \nu(x,z)$, one may conclude, that together with z_k for some k, $-z_k$ also satisfies characteristic equation (9). Furthermore, using properties of Fourier transform one may prove, that $\overline{u}(-z_k) = \overline{\overline{u}(z_k)}$ and $M_k(-z_k) = \overline{M_k(z_k)}$, where the line over expressions means their complex adjoint, therefore investigation of systems (8) and (11) may be limited only for roots z_k, $k \in \mathbb{N}$.

Remark 2.3. In the case when all roots of characteristic equation (9) are real: $\gamma_k = 0$, $k \in \mathbb{N}$, we obtain

$$\begin{aligned} M_{pk} = \sum_{j,s=1}^{m,n} P_{js}\Big[K_1(x_j,\sigma_k)\{\sin;-\cos\}\sigma_k(t_j-\tau) + K_2(x_j,\sigma_k)\{\cos;\sin\}\sigma_k(t_j-\tau)\Big] + \\ + \{\cos;-\sin\}(\sigma_k\tau)\int_0^l \Big[K_1(x,\sigma_k)g_2(x,\sigma_k) + K_2(x,\sigma_k)g_1(x,\sigma_k)\Big]dx - \\ - \{\sin;\cos\}(\sigma_k\tau)\int_0^l \Big[K_1(x,\sigma_k)g_1(x,\sigma_k) - K_2(x,\sigma_k)g_2(x,\sigma_k)\Big]dx, \ p \in \{1,2\}, \end{aligned}$$

$$K_1(x,\sigma) = \mathrm{Re}K(l,x,\sigma) = \frac{\cos(\lambda(l,\sigma) - \lambda(x,\sigma)) - \cos(\mu(l,\sigma) - \mu(x,\sigma))}{\lambda'(x,\sigma) - \mu'(x,\sigma)},$$

$$K_2(x,\sigma) = \mathrm{Im}K(l,x,\sigma) = \frac{\sin(\lambda(l,\sigma) - \lambda(x,\sigma)) - \sin(\mu(l,\sigma) - \mu(x,\sigma))}{\lambda'(x,\sigma) - \mu'(x,\sigma)}.$$

Thus, the problem of optimal control under consideration was reduced to minimization procedure of functional (5) with respect to $u \in U$ under integral constraints (11). Solution of reduced problem can be constructed by different rigorous techniques or effective numerical methods of non–linear programming. Nevertheless, as kernels of equalities (11) are bounded in integration interval, it is convenient to consider it as a system of moments problem with respect to desired function [6]. In order to proceed to its solution, we will deal with truncated part of countable system (11) for some natural N. The convergence of solutions of finite system of moments problem to a solution of infinite one is investigated as it is done in [10]. Using well-known technique for solving moments problem under integral constraints, outlined in [6], solution of the system (11) can be obtained explicitly:

$$u_N^o(t) = \sum_{q=1}^{Q} u_{Nq}^o \delta(t - t_q^o); \; t \in [t_0, T - \tau], \tag{12}$$

where intensities of control actions u_{Nq}^o are constrained by conditions

$$\operatorname{sgn} u_{Nq}^o = \operatorname{sgn} h_N^o(t_q^o), \; q \in \{1, \ldots, Q\},$$

where

$$h_N^o(t) = \sum_{k=1}^{N} e^{-\gamma_k t} \left[\Lambda_{1k}^o \cos(\sigma_k t) + \Lambda_{2k}^o \sin(\sigma_k t)\right]; \; t \in [t_0, T - \tau]$$

and are determined from the following system of equations:

$$\sum_{q=1}^{Q} u_{Nq}^o e^{-\gamma_k t_q^o} \cos(\sigma_k t_q^o) = M_{1k}, \sum_{q=1}^{Q} u_{Nq}^o e^{-\gamma_k t_q^o} \sin(\sigma_k t_q^o) = M_{2k}; \; k \in \{1, \ldots, N\},$$

and the moments of control impacts applications are determined from maximum condition

$$\sup_{t \in [t_0, T-\tau]} \left| \sum_{k=1}^{N} e^{-\gamma_k t} \Big[\Lambda_{1k}^o \cos(\sigma_k t) + \Lambda_{2k}^o \sin(\sigma_k t)\Big] \right| = \Big[\sum_{q=1}^{Q} |u_{Nq}^o|\Big]^{-1}, \tag{13}$$

Resolving coefficients Λ_{1k}^o and Λ_{2k}^o are determined from the following problem of conditional extremum:

$$\sum_{k=1}^{N} e^{-\gamma_k t_q^o} \left[\Lambda_{1k} \cos(\sigma_k t_q^o) + \Lambda_{2k} \sin(\sigma_k t_q^o)\right] \underset{\{\Lambda_{1k}, \Lambda_{2k}\}}{\longrightarrow} \min, \; \sum_{k=1}^{N} \left[\Lambda_{1k} M_{1k} + \Lambda_{2k} M_{2k}\right] = 1.$$

The number Q is determined from condition that inclusions $\{t_q^o\}_{q=1}^{Q} \subset [t_0, T - \tau]$ must be satisfied for any fixed N.

Answer to the second part of the problem concerning necessary and sufficient conditions for existence of admissible control function, which ensure for the string the end state (4), gives the following

Theorem 2.4. *System (1)–(2) is fully controllable in* $\mathrm{L}^1[t_0, T]$ *if and only if the quantity*

$$\rho_N^o = \sum_{q=1}^{Q} |u_{Nq}^o| \tag{14}$$

differs from zero for all N.

Proof. According to [6, 10], for solvability of system of moments problem (11) it is necessary and sufficient, that the norm of $h_N^o(t)$ functions in terms of function space, adjoint to admissible control function space, be bounded and non–zero for all natural N. As in this case the space of Lebesgue measurable functions $\mathrm{L}^1[t_0, T-\tau]$ is taken as admissible control function space, which is adjoint to the space $\mathrm{L}^\infty[t_0, T-\tau]$ with norm $||h_N^o(t)|| = \sup |h_N^o(t)| \equiv [\rho_N^o]^{-1}$, then taking into account (12) and related maximum condition (13) we will obtain (14). $\square$

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