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SOME PROPERTIES OF BOUNDARY VALUE PROBLEMS FOR BESSEL'S EQUATION

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Some properties of systems of the Bessel functions of negative order less than -1 generated by one boundary value problem are studied.

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Досліджено деякі властивості систем функцій Бесселя з від'ємним індексом меншим за -1, які виникають в одній крайовій задачі.

Let $\nu > 0$ be a non-integer number, $\overline{0;\nu} = \mathbb{Z} \cap [0;\nu]$, $p \in \mathbb{R}$, and $L_2\left((0;1); x^p dx\right)$ be the Hilbert space of functions $f : (0;1) \to \mathbb{C}$ such that the function $t^{p/2} f(t)$ belongs to the space $L_2(0;1)$; the inner product and the norm in $L_2\left((0;1); x^p dx\right)$ are given respectively by $\langle f_1; f_2 \rangle = \int_0^1 t^p f_1(t)\overline{f_2(t)}dt$ and $\|f\| = \sqrt{\int_0^1 t^p |f(t)|^2 dt}$. Further, let J_ν be Bessel's function of order ν. We study the approximation properties of the system $\{J_{-\nu}(\rho_j x)\}$ for some sequence (ρ_j) tending to infinity. Such a system arises, for example, when considering the boundary value problem

$$-f'' + \frac{\nu^2 - 1/4}{x^2} f = \lambda f, \tag{1}$$

$$f(1) = 0, \tag{2}$$

$$\exists c_k \in \mathbb{C} : \qquad f(x) = \sum_{k \in \overline{0;\nu}} c_k x^{-\nu+2k+1/2} + o(x^{\nu+1/2}), \quad x \to 0+. \tag{3}$$

Indeed, one easily derives the following result:

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Proposition 1. *The boundary value problem (1)–(3) has a countable set of eigenvalues* $\{\lambda_j : j \in \mathbb{N}\}$*, and* $\lambda_j = \rho_j^2$*, where* ρ_j *are the zeros of the Bessel function* $J_{-\nu}$*; moreover,* $v_j(x) = \rho_j^\nu \sqrt{\pi x/2} J_{-\nu}(\rho_j x)$ *are the corresponding eigenfunctions.*

Proof. For $\lambda \neq 0$, equation (1) has two linearly independent solutions given by $u(x) = \rho^{-\nu-1/2}\sqrt{\pi\rho x/2} J_\nu(\rho x)$ and $v(x) = \chi_\nu \rho^{\nu-1/2}\sqrt{\pi\rho x/2} J_{-\nu}(\rho x)$, where $\rho = \sqrt{\lambda}$ and $\chi_\nu = -1/\sin(\nu\pi)$. Using the asymptotic behaviour of the Bessel functions, we find that

$$u(x) = \sqrt{\pi/2}\frac{x^{\nu+1/2}}{2^\nu\Gamma(\nu+1)} + o(x^{\nu+1/2}), \quad x \to 0+,$$

$$v(x) = \chi_\nu\sqrt{\pi/2}\sum_{k\in\overline{0;\nu}} \frac{(-1)^k\rho^{2k}x^{-\nu+2k+1/2}}{2^{-\nu+2k}k!\Gamma(-\nu+k+1)} + O(x^{-\nu+2[\nu]+5/2}), \quad x \to 0+ .$$

Therefore the general solution $\alpha u(x) + \beta v(x)$ of (1) satisfies the condition (3) if and only if $\alpha = 0$. Recalling (2), we conclude that the eigenfunctions of the boundary value problem (1)–(3) are $v_j(x) = \rho_j^\nu\sqrt{\pi x/2}J_{-\nu}(\rho_j x)$, and the corresponding eigenvalues are $\{\lambda_j : j \in \mathbb{N}\}$, where $\lambda_j = \rho_j^2$ and ρ_j are the zeros of the Bessel function $J_{-\nu}$. The proposition is proved. □

The function $J_{-3/2}$ has ([1]) an infinite set $\{\rho_j : j \in \mathbb{Z}\backslash\{0\}\}$ of zeros; moreover, the zeros ρ_1 and $\rho_{-1} = \overline{\rho_1} = -\rho_1$ are purely imaginary, ρ_j with $j \in \mathbb{N}\backslash\{1\}$ are positive, and $\rho_{-j} := -\rho_j$, $j \in \mathbb{N}\backslash\{1\}$, are negative. In [2]–[4] the following results were proved.

Theorem A. *Let* ρ_j *be the zeros of the function* $J_{-3/2}$*. Then the system* $\{\rho_j\sqrt{\rho_j x}J_{-3/2}(\rho_j x) : j \in \mathbb{N}\backslash\{1\}\}$ *is complete in the space* $L_2((0;1); x^2dx)$*, has in this space a biorthogonal system* $\{\gamma_k : k \in \mathbb{N}\backslash\{1\}\}$*,*

$$\overline{\gamma_k}(x) := \frac{\pi(\rho_k\sqrt{\rho_k x}J_{-3/2}(\rho_k x) - \rho_1\sqrt{\rho_1 x}J_{-3/2}(\rho_1 x))}{x^2\rho_k^2\cos^2\rho_k},$$

and is not a basis of this space.

Theorem B. *For* $\nu = 3/2$ *the boundary value problem (1)–(3) has a countable set of eigenvalues* $\{\lambda_j : j \in \mathbb{N}\}$*; moreover,* $\lambda_j = \rho_j^2$*, where* ρ_j *are the zeros of the Bessel function* $J_{-3/2}$*. In particular, all eigenvalues are real,* λ_1 *is negative, and* λ_j *are positive for* $j > 1$*. The corresponding eigenfunctions* $\rho_j\sqrt{\rho_j x}J_{-3/2}(\rho_j x)$ *enjoy the properties described in Theorem A.*

Similar results can be obtained for $\nu = 5/2$. Their extension to arbitrary ν leads to some difficulties. In this connection we will prove here the following proposition.

Theorem 1. *Let* $z \in \mathbb{C}$ *and let* ρ_j *be the zeros of the function* $J_{-\nu}$ *with* $\nu = m+1/2$*,* $m \in \mathbb{N}$*. Then*

$$\int_0^1 \cos(zt)\frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}}dt = (-1)^{m+1}\sqrt{\frac{\pi}{2}}\rho_j^{1+\nu}J_{-\nu+1}(\rho_j)\left(\frac{1}{z}\frac{d}{dz}\right)^m \frac{J_{-\nu}(z)}{(z^2-\rho_j^2)z^{-\nu}},$$

where $(z^{-1}d/dz)^m$ *is the* m*-th power of the differential operator* $z^{-1}d/dz$*.*

Proof. The functions $f(t) = J_{\pm\nu}(zt)$ are the solutions of the equation $f'' + f'/t - \nu^2 f/t^2 = -z^2 f$. Hence

$$\frac{d}{dt}\left(t\frac{dJ_{-\nu}(zt)}{dt}\right) + \left(z^2 t - \frac{\nu^2}{t}\right) J_{-\nu}(zt) = 0,$$
$$\frac{d}{dt}\left(t\frac{dJ_{-\nu}(\rho_j t)}{dt}\right) + \left(\rho_j^2 t - \frac{\nu^2}{t}\right) J_{-\nu}(\rho_j t) = 0.$$

Multiply the first equality by $J_{-\nu}(\rho_j t)$, the second by $J_{-\nu}(zt)$ and subtract. Then

$$\frac{d}{dt}\left(t\left(\frac{dJ_{-\nu}(zt)}{dt}J_{-\nu}(\rho_j t) - \frac{dJ_{-\nu}(\rho_j t)}{dt}J_{-\nu}(zt)\right)\right) = -(z^2-\rho_j^2)tJ_{-\nu}(zt)J_{-\nu}(\rho_j t)$$

or, in turn,

$$\rho_j^{-\nu}t^{-2\nu+1}\frac{J_{-\nu}(zt)}{(zt)^{-\nu}}\frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} = \frac{d}{dt}\left(t\frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2-\rho_j^2)z^{-\nu}}\right),$$

and

$$\rho_j^{-\nu}t^{-2\nu+1}\frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^m \frac{J_{-\nu}(zt)}{(zt)^{-\nu}} =$$
$$= \left(\frac{1}{z}\frac{\partial}{\partial z}\right)^m \frac{\partial}{\partial t}\left(t\frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2-\rho_j^2)z^{-\nu}}\right).$$

Using the equality [5, p.56]

$$\frac{d}{dz}\frac{J_{-\nu}(z)}{z^{-\nu}} = -z^{\nu}J_{-\nu+1}(z),$$

we obtain

$$\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^m \frac{J_{-\nu}(zt)}{(zt)^{-\nu}} = (-1)^m t^{2m}\frac{J_{-\nu+m}(zt)}{(zt)^{-\nu+m}}.$$

Therefore

$$(-1)^m\rho_j^{-\nu}\frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}}t^{-2\nu+1+2m}\frac{J_{-\nu+m}(zt)}{(zt)^{-\nu+m}} =$$
$$= \left(\frac{1}{z}\frac{\partial}{\partial z}\right)^m \frac{\partial}{\partial t}\left(t\frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2-\rho_j^2)z^{-\nu}}\right).$$

Since $-\nu + m = -1/2$ and $t^{1/2}J_{-1/2}(t) = \sqrt{2/\pi}\cos t$, we have

$$(-1)^m\rho_j^{-\nu}\sqrt{\frac{2}{\pi}}\frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}}\cos(zt) =$$
$$= \left(\frac{1}{z}\frac{\partial}{\partial z}\right)^m \frac{\partial}{\partial t}\left(t\frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2-\rho_j^2)z^{-\nu}}\right) =$$
$$= \frac{\partial}{\partial t}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^m \left(t\frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2-\rho_j^2)z^{-\nu}}\right).$$

Hence

$$(-1)^m \rho_j^{-\nu} \sqrt{\frac{2}{\pi}} \int_0^1 \cos(zt) \frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} dt =$$

$$= \int_0^1 \frac{\partial}{\partial t} \left(\frac{1}{z} \frac{\partial}{\partial z} \right)^m \left(t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right) dt =$$

$$= \left(\frac{1}{z} \frac{\partial}{\partial z} \right)^m t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \Bigg|_0^1 =$$

$$= \rho_j J'_{-\nu}(\rho_j) \left(\frac{1}{z} \frac{d}{dz} \right)^m \frac{J_{-\nu}(z)}{(z^2 - \rho_j^2) z^{-\nu}}.$$

But ([5, p.56]) $tJ'_{-\nu}(t) + \nu J_{-\nu}(t) = -tJ_{-\nu+1}(t)$. Then $J'_{-\nu}(\rho_j) = -J_{-\nu+1}(\rho_j)$, and we obtain the desired result. The theorem is proved. □

Theorem 1 for $\nu = 3/2$ was used to prove completeness of the corresponding systems of eigenfunction in Theorem B.

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