ON TRANSCENDENCE OF MODULUS OF JACOBI ELLIPTIC FUNCTIONS

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Ivan Franko Lviv National University,
1 Universytetska Str., Lviv 79000, Ukraine

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Let $\mathrm{sn}_1 z, \mathrm{sn}_2 z$ be the Jacobi elliptic functions, $\varkappa_1, \varkappa_2$ the moduli of these elliptic functions, $0 < \varkappa_1^2 < 1$, $0 < \varkappa_2^2 < 1$, τ_1, τ_2 the values of modular variable, $\theta_3(\tau_1), \theta_3(\tau_2)$ the theta constants. In this paper it is shown that there exists a transcendental number among $\varkappa_1, \varkappa_2, \theta_3(\tau_1), \theta_3(\tau_2)$, if τ_1/τ_2 is irrational.

INTRODUCTION

Let $\mathrm{sn}_1 z$, $\mathrm{sn}_2 z$ be the Jacobi elliptic functions, $\varkappa_j$ the modulus of $\mathrm{sn}_j z$, $j = 1, 2$. They are determined by the values of modular variable τ_j respectively [7]. We use the notation [7] for the theta functions: $\theta_i(z, \tau_j)$ is the theta function ($i = 2, 3, 4, j = 1, 2$) of z determined by the values of modular variable τ_j; $\theta_{i,j}$ are the theta constants, $\theta_{i,j} = \theta_i(0, \tau_j)$.

We refer to [1, 4, 6] for some information on arithmetic properties of numbers related to elliptic functions. In this paper we obtain the following result.

Theorem 1. *Let $0 < \varkappa_1^2 < 1$, $0 < \varkappa_2^2 < 1$, and τ_1/τ_2 is irrational. Then at least one of the numbers $\varkappa_1$, $\varkappa_2$, $\theta_{3,1}$, $\theta_{3,2}$ is transcendental.*

An analogous result for the theta functions is obtained in [4].

As for the theta functions Jacobi, the elliptic functions $\mathrm{sn}\, z, \mathrm{cn}\, z$ и $\mathrm{dn}\, z$ are related. In particular, they satisfy the following relations

$$\varkappa_j^2 = \frac{\theta_{2,j}^4}{\theta_{3,j}^4}, \quad \mathrm{sn}_j z = \frac{\theta_{3,j}\theta_1(v, \tau_j)}{\theta_{2,j}\theta_4(v, \tau_j)}, \quad v = \frac{z}{\pi\theta_{3,j}^2}, \quad j = 1, 2, \tag{1}$$

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$$|\theta_i(z,\tau)| \leqslant \exp(\gamma|z|^2). \tag{2}$$

Polynomials of $\varkappa$ with integer coefficients are the coefficients of expansions of Jacobi functions into the Taylor series.

Lemma 1. *The following conditions take place:*

$$\operatorname{sn} z = \sum_{j=0}^{\infty} A_{1,2j+1}(\varkappa)\frac{z^{2j+1}}{(2j+1)!}, \quad \operatorname{cn} z = \sum_{j=0}^{\infty} A_{2,2j}(\varkappa)\frac{z^{2j}}{(2j)!}, \tag{3}$$

$$\operatorname{dn} z = \sum_{j=0}^{\infty} A_{3,2j}(\varkappa)\frac{z^{2j}}{(2j)!},$$

where

$$\begin{gathered} A_{1,2j+1}(\varkappa) \ll (2j+1)!\varkappa^{2j},\ A_{2,2j}(\varkappa) \ll (2j)!\varkappa^{2j-2}, \\ A_{3,2j}(\varkappa) \ll (2j)!\varkappa^{2j}. \end{gathered} \tag{4}$$

Proof. We proceed by induction on j. For $j=0$ we have $A_{1,1}(\varkappa)=1$, $A_{2,0}(\varkappa)=1$, $A_{3,0}(\varkappa)=1$; for $j=1$ we have $A_{1,3}(\varkappa)=-(1+\varkappa^2)$, $A_{2,2}(\varkappa)=-1$, $A_{3,2}(\varkappa)=-\varkappa^2$. Suppose that (4) holds for all coefficients in (3) with $j \leqslant j_0$. Conditions (1) imply

$$A_{1,2j_0+1}(\varkappa) = \sum_{k=0}^{j_0} \frac{(2j_0)!}{(2k)!(2(j_0-k))!} A_{2,2k}(\varkappa)A_{3,2(j_0-k)}(\varkappa). \tag{5}$$

The required estimate follows from the induction hypothesis.

The proof is complete.

Lemma 2. *For all positive integer t_1, t_2 there exist polynomials $B_{r,t_1,t_2} \in \mathbb{Z}[\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}]$ such that*

$$\operatorname{sn}_1^{t_1}(z)\operatorname{sn}_2^{t_2}(z) = \sum_{r=0}^{\infty} B_{r,t_1,t_2}(\varkappa_1,\varkappa_2,\theta_{3,1},\theta_{3,2})\frac{z^r}{r!} \tag{6}$$

and

$$\begin{gathered} B_{r,t_1,t_2}(\varkappa_1,\varkappa_2,\theta_{3,1},\theta_{3,2}) \ll \\ \ll r!t_1!t_2!2^{\frac{r+t_1+t_2}{2}}(\varkappa_1+\varkappa_2+\theta_{3,1}+\theta_{3,2})^{4r-2t_1-2t_2}. \end{gathered} \tag{7}$$

Proof. If $r < t_1 + t_2$ then let $B_{r,t_1,t_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) = 0$. Lemma 1 applied to $\mathrm{sn}_1(\theta_{3,1}^2 z)$, $\mathrm{sn}_2(\theta_{3,2}^2 z)$, and (3) imply

$$\sum_{r=0}^{\infty} B_{r,t_1,t_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) \frac{z^r}{r!} =$$
$$= \left(\sum_{j=0}^{\infty} A_{1,2j+1}(\varkappa_1) \frac{(\theta_{3,1}^2 z)^{2j+1}}{(2j+1)!} \right)^{t_1} \left(\sum_{j=0}^{\infty} A_{1,2j+1}(\varkappa_2) \frac{(\theta_{3,2}^2 z)^{2j+1}}{(2j+1)!} \right)^{t_2}. \quad (8)$$

For $r \geqslant t_1 + t_2$ let us compare the coefficients at z^r in the left and right parts of (8):

$$\frac{B_{r,t_1,t_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})}{r!} =$$
$$= \sum\nolimits_1 t_1! t_2! \prod_{m=0}^{n} \frac{A_{1,2m+1}^{j_m}(\varkappa_1) A_{1,2m+1}^{k_m}(\varkappa_2) (\theta_{3,1})^{2r_1} (\theta_{3,2})^{2r_2}}{j_m! k_m! ((2m+1)!)^{j_m+k_m}}, \quad (9)$$

where the sum $\sum_1$ is taken over all nonnegative integers j_m, k_m such that $n = [r/2]$, $j_0 + \ldots + j_n = t_1$, $k_0 + \ldots + k_n = t_2$, $j_0 + 3j_1 + \ldots + (2n+1)j_n = r_1$, $k_0 + 3k_1 + \ldots + (2n+1)k_n = r_2$, $r_1 + r_2 = r$. The following estimate can be obtained from (4), (9):

$$B_{r,t_1,t_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) \ll$$
$$\ll r! t_1! t_2! (\varkappa_1 + \varkappa_2 + \theta_{3,1} + \theta_{3,2})^{4r-2t_1-2t_2} \sum\nolimits_1 \prod_{m=0}^{n} \frac{1}{j_m! k_m!} \ll$$
$$\ll r! t_1! t_2! (\varkappa_1 + \varkappa_2 + \theta_{3,1} + \theta_{3,2})^{4r-2t_1-2t_2} 2^{\frac{r+t_1+t_2}{2}}. \quad (10)$$

The proof is complete.

Lemma 3. *For every sufficiently large integer N there exist polynomials $C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})$ from $\mathbb{Z}[x_1, x_2, x_3, x_4]$, $0 \leqslant k_1, k_2 \leqslant K$, $K = [4\sqrt{N}]$, such that the function*

$$F(z) = \sum_{k_1,k_2=0}^{K} C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) \, \mathrm{sn}_1^{k_1}(\theta_{3,1}^2 z) \, \mathrm{sn}_2^{k_2}(\theta_{3,2}^2 z) \quad (11)$$

satisfy $F(z) \not\equiv 0$ and

$$\mathrm{ord}_{z=0} F \geqslant N, \ \deg C_{k_1,k_2} \leqslant N + 2k_1 + 2k_2, \ \ln |C_{k_1,k_2}| \leqslant 2N \ln N. \quad (12)$$

Proof. From Lemma 2 and the choice of the parameter K it follows that the following estimate is true:

$$|B_{r,k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})| \leqslant N! C_1^N, \quad r \leqslant N. \tag{13}$$

Consider the set of polynomials $D_{N,k_1,k_2}(x_1, x_2, x_3, x_4)$ with undefined coefficients $a_{l_1,l_2,l_3,l_4}(k_1, k_2) \in \mathbb{Z}$,

$$D_{N,k_1,k_2}(x_1, x_2, x_3, x_4) = \sum\nolimits_2 a_{l_1,l_2,l_3,l_4}(k_1, k_2) x_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4}, \tag{14}$$

where the sum $\sum_2$ is taken over all nonnegative integers l_1, l_2, l_3, l_4 such that $l_1 + l_2 + l_3 + l_4 = N + 2k_1 + 2k_2$.

Choose $a_{l_1,l_2,l_3,l_4}(k_1, k_2) \in \mathbb{Z},\ a_{l_1,l_2,l_3,l_4}(k_1, k_2) \not\equiv 0$, so that for $0 \leqslant r < N$ the following relations take place:

$$\sum_{k_1,k_2=0}^{K} D_{N,k_1,k_2}(x_1, x_2, x_3, x_4) B_{r,k_1,k_2}(x_1, x_2, x_3, x_4) \equiv 0. \tag{15}$$

Applying to system (15) Siegel's Lemma (see, for example, [5]), we obtain that exist there polynomials, not of all equal to zero, $a_{l_1,l_2,l_3,l_4}(k_1, k_2) \in \mathbb{Z}$ such that (15) hold and

$$|a_{l_1,l_2,l_3,l_4}(k_1, k_2)| < N^N C_2^N, \quad a_{l_1,l_2,l_3,l_4}(k_1, k_2) \not\equiv 0. \tag{16}$$

Applying the constructed polynomials $D_{N,k_1,k_2}(x_1, x_2, x_3, x_4)$ we find polynomials $C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})$ in (11) such that $C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) \not\equiv 0$.

We denote

$$\mathcal{D}_{s_1,s_2,s_3,s_4} = \frac{1}{s_1! s_2! s_3! s_4!} \frac{\partial^s}{\partial^{s_1} x_1 \partial^{s_2} x_2 \partial^{s_3} x_3 \partial^{s_4} x_4}, \quad s_1 + s_2 + s_3 + s_4 = s. \tag{17}$$

Let s be the minimal integer with $0 \leqslant s < N$ such that there exist $s_1, s_2, s_3, s_4,\ s_1 + s_2 + s_3 + s_4 = s$ satisfying the conditions

$$\mathcal{D}_{s_1,s_2,s_3,s_4} D_{N,k_1,k_2}(x_1, x_2, x_3, x_4)|_{(x_1,x_2,x_3,x_4)=(\varkappa_1,\varkappa_2,\theta_{3,1},\theta_{3,2})} \neq 0 \tag{18}$$

for some $D_{N,k_1,k_2}(x_1, x_2, x_3, x_4)$. Let

$$\begin{aligned} &C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) = \\ &= \mathcal{D}_{s_1,s_2,s_3,s_4} D_{N,k_1,k_2}(x_1, x_2, x_3, x_4)|_{(x_1,x_2,x_3,x_4)=(\varkappa_1,\varkappa_2,\theta_{3,1},\theta_{3,2})}. \end{aligned} \tag{19}$$

Apply the operator $\mathcal{D}_{s_1,s_2,s_3,s_4}$ to the left part of (15). By the choice of s we see that for all r, $0 \leqslant r < N$, (19) implies

$$E_r(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) = \\ = \sum_{k_1,k_2=0}^{K} C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) B_{r,k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) = 0. \quad (20)$$

From (6), (11), (19), (20) it follows that

$$F^{(r)}(0) = E_r(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) = 0, \quad 0 \leqslant r < N. \quad (21)$$

Conditions of Theorem 1 imply algebraic independence of the functions $\mathrm{sn}_1(\theta_{3,1}^2 z)$, $\mathrm{sn}_2(\theta_{3,2}^2 z)$, therefore $F(z) \not\equiv 0$. Let us estimate the coefficients $C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})$. From (14), (16) we obtain

$$D_{N,k_1,k_2}(x_1, x_2, x_3, x_4) \ll N! C_3^N (x_1 + x_2 + x_3 + x_4)^{N+2k_1+2k_2}. \quad (22)$$

It follows from (19), (22) that

$$C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) \ll N! C_4^N (\varkappa_1 + \varkappa_2 + \theta_{3,1} + \theta_{3,2})^{N-s+2k_1+2k_2}. \quad (23)$$

The proof is complete.

Let T be minimal integer such that $F^{(T)}(0) \neq 0$. Then $T \geqslant N$.

Lemma 4. *There exists a polynomial $R_T \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ such that*

$$F^{(T)}(0) = R_T(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}), \quad (24)$$

$$\deg R_T(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) \leqslant 5T, \quad \ln |R_T(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})| \leqslant C_5 T \ln T, \quad (25)$$

$$0 < |R_T(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})| < \exp(-C_6 T \sqrt{T}). \quad (26)$$

Proof. It follows from (6), (11), (21) and the definition of T that

$$F^{(T)}(0) = \sum_{k_1,k_2=0}^{K} C_{k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}) B_{T,k_1,k_2}(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}). \quad (27)$$

Let

$$R_T(x_1, x_2, x_3, x_4) = \sum_{k_1,k_2=0}^{K} C_{k_1,k_2}(x_1, x_2, x_3, x_4) B_{T,k_1,k_2}(x_1, x_2, x_3, x_4). \quad (28)$$

From (7) it follows that Lemma 3 and the choice K imply estimates (25). From the definition of T it follows that $|R_T(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})| > 0$.

Consider the function

$$G(z) = F(z)\theta_4^K\left(\frac{z}{\pi}, \tau_1\right)\theta_4^K\left(\frac{z}{\pi}, \tau_2\right). \tag{29}$$

From (1), (12), (29) and the properties of $\theta_4(z, \tau)$ it follows that $G(z)$ is an entire periodical function with period 2π and zeros in $2\pi n$. The choice of T implies that the order of zeros is equal to T, therefore the function

$$H(z) = \frac{G(z)}{\prod_{|n|\leqslant M}(z - 2\pi n)^T}, \tag{30}$$

where $M = [C_7\sqrt{T}]$, is entire. Thus

$$|H(0)| \leqslant \max_{|z|=4\pi M} |H(z)|. \tag{31}$$

From (2), (13), (23), (29) and the choice of T, N for $|z| \leqslant 4\pi M$ it follows that

$$|G(z)| \leqslant \exp(C_8 T\sqrt{T}) \tag{32}$$

For $|z| = 4\pi M$ we have

$$\prod_{|n|\leqslant M}(z - 2\pi n)^T \geqslant (M!)^{2T}(2\pi)^{2MT}. \tag{33}$$

From (12), (29), (30) and the choice of T it follows that

$$|F^{(T)}(0)| = T!(M!)^{2T}(2\pi)^{2MT}(\theta_4(0, \tau_1)\theta_4(0, \tau_2))^{-K}H(0). \tag{34}$$

From (30) – (34) it follows that

$$|F^{(T)}(0)| < \exp(-C_9 T\sqrt{T}). \tag{35}$$

It follows from (24) and (35) that

$$|R_T(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})| < \exp(-C_{10} T\sqrt{T}). \tag{36}$$

The proof is complete.

Suppose that Theorem 1 is not true and $\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2}$ are algebraic numbers. Then (27) implies that $F^{(T)}(0)$ is the value of a polynomial with algebraic coefficients at the algebraic point $(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})$. Applying Liouville's Theorem [5], we obtain

$$|R_T(\varkappa_1, \varkappa_2, \theta_{3,1}, \theta_{3,2})| > \exp(-C_{11} T \ln T),$$

which contradicts to (36) and the proof of Theorem 1 is complete.

[1] *Fel'dman N.I., Nesterenko Yu.V.* Transcendental Numbers. – Springer, 1998.

[2] *Kholyavka Ya.M.* On a measure of algebraic independence of values of Jacobi elliptic functions // Fundamental and applied mathematics. – **11**, No. 6. – P. 209–219.

[3] *Masser D.* Elliptic functions and transcendence // Lect. Notes Math. – 1975., V. 437. – P. 1–143.

[4] *Nesterenko Yu.V.* On arithmetic properties of values of theta-constants // Fundamental and applied mathematics. – **11**, No. 6. – P. 95–122.

[5] *Shidlovskij A.B.* Transcendental Numbers. – Walter de Grueter. – Berlin–New York, 1987.

[6] *Waldschmidt M.* Elliptic functions and transcendence // http://www.math.jussieu.fr/ ~miw/

[7] *Whittaker E.T., Watson G.N.* A Course of Modern Analysis. – Cambridge, 1927.

ПРО ТРАНСЦЕНДЕНТНІСТЬ МОДУЛІВ ЕЛІПТИЧНИХ ФУНКЦІЙ ЯКОБІ

Ярослав ХОЛЯВКА

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів 79000, Україна

Нехай $\mathrm{sn}_1 z, \mathrm{sn}_2 z$ — еліптичні функції Якобі, $\varkappa_1, \varkappa_2$ — модулі цих функцій, $0 < \varkappa_1^2 < 1$, $0 < \varkappa_2^2 < 1$, τ_1, τ_2 — значення модулярної змінної; $\theta_3(\tau_1), \theta_3(\tau_2)$ — тета-константи. Доведено існування трансцендентного числа серед $\varkappa_1, \varkappa_2, \theta_3(\tau_1)$ та $\theta_3(\tau_2)$, якщо τ_1/τ_2 — ірраціональне.