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S. V. Sharyn

GÂTEAUX DIFFERENTIABILITY OF POLYNOMIAL TEST AND GENERALIZED FUNCTIONS

Let $\mathcal{S}_+$ and $\mathcal{S}'_+$ be the Schwartz spaces of rapidly decreasing functions and tempered distributions on $\mathbb{R}_+$, respectively. Let $\mathrm{P}(\mathcal{S}'_+)$ be the space of continuous polynomials over $\mathcal{S}'_+$ and $\mathrm{P}'(\mathcal{S}'_+)$ be its strong dual. These spaces have representations in the form of Fock type spaces $\Gamma(\mathcal{S}_+) := \bigoplus_{n\in\mathbb{Z}_+} (\otimes^n_{s,\mathrm{p}} \mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+) := \times_{n\in\mathbb{Z}_+} (\otimes^n_{s,\mathrm{p}} \mathcal{S}'_+)$, respectively. In the paper the Gâteaux differentiability of the elements of the spaces $\mathrm{P}(\mathcal{S}'_+)$, $\mathrm{P}'(\mathcal{S}'_+)$, $\Gamma(\mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+)$ is investigated. It is established connection of Gâteaux derivative with the creation and annihilation operators on the Fock type spaces as well as with differentiations on $\Gamma(\mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+)$.

Introduction. Many problems in applied mathematics can be modelled and solved within framework of the theory of linear continuous functionals (generalized functions, distributions) [1]. This theory has a disadvantage: in general the space of distributions has no an algebraic structure. However, some problems (e.g., in quantum field theory) require such a structure, moreover, nonlinear generalization of a distribution concept is needed [4].

In the previous papers [2, 9] it is introduced the theory of polynomial (nonlinear) distributions and ultradistributions. With the help of a tensor representation of such spaces it is described their algebraic structure. Differentiations and semigroup of polynomial shifts are investigated as well.

Recent developments of white noise calculus [6, 7, 10] consider any operator on a Fock space as a function of annihilation and creation operators, which emphasizes their importance. These two operators are referred to as the quantum white noise.

The aim of this work is to establish the Gâteaux differentiability of polynomial test $P \in \mathrm{P}(\mathcal{S}'_+)$ and generalized functions $F \in \mathrm{P}'(\mathcal{S}'_+)$ and of elements of corresponding Fock type spaces $\Gamma(\mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+)$. Explicit formulas for translation operator and Gâteaux derivative are presented. The another purpose is to establish the connection of Gâteaux derivative with the quantum white noise on the spaces $\Gamma(\mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+)$ as well as with differentiations, investigated earlier [2, 9]. Note that our results are in accord with the analogous ones of other authors [6, 7]. The present work can be used in infinite dimensional stochastic analysis (see, e.g., [3, 8] and references therein).

1. Preliminaries and notations. Let $\mathcal{L}(X,Y)$ be the space of all linear continuous operators from X to Y and $\mathcal{L}(X) := \mathcal{L}(X,X)$, where X and Y are locally convex topological vector spaces. We endow $\mathcal{L}(X)$ with the locally convex topology of uniform convergence on the bounded subsets of X. Everywhere $X' := \mathcal{L}(X,\mathbb{C})$ denotes strong dual space of X. For any $f \in X'$, $x \in X$ we will denote by $\langle f, x\rangle$ the pairing between elements of respective spaces.

In the sequel, $\otimes_{s,\mathrm{p}}$ denotes a completion of symmetric tensor product $\otimes_s$ in the projective tensor locally convex topology and $\otimes^n_{s,\mathrm{p}} X$ denotes symmetric projective tensor product of n copies of a space X. Let by definition $\otimes_{s,\mathrm{p}}^0 X := \mathbb{C}$. We denote $x^{\otimes n} := \underbrace{x \otimes \ldots \otimes x}_{n} \in \otimes_{s,\mathrm{p}}^n X$ and $x^{\otimes 0} = 1 \in \mathbb{C}$ for any $x \in X$.

Schwartz spaces. Let $\mathcal{S} := \mathcal{S}(\mathbb{R})$ and $\mathcal{S}' := \mathcal{S}'(\mathbb{R})$ stand for the classical Schwartz spaces of rapidly decreasing functions and tempered distributions, respectively.

Denote by $\mathcal{S}_+ := \mathcal{S}(\mathbb{R}_+)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}_+ := [0, +\infty)$ and by $\mathcal{S}'_+$ the distribution space that is dual of $\mathcal{S}_+$. It is easy to see that elements of the space $\mathcal{S}'_+$ may be treated as distributions from $\mathcal{S}'$, vanishing on $(-\infty, 0)$, therefore, $\mathcal{S}'_+$ is a subspace of $\mathcal{S}'$.

Let δ_t be the Dirac delta-functional concentrated at a point $t \in \mathbb{R}_+$. It is well known that the space $\mathcal{S}'_+$ is a topological algebra with the unit $\delta := \delta_0$ relative to the convolution, defined by

$$\langle f * g, \varphi \rangle = \langle f(s), \omega(s) \langle g(t), v(t)\varphi(s+t) \rangle \rangle, \qquad f, g \in \mathcal{S}'_+, \qquad \varphi \in \mathcal{S}_+,$$

where the functions $\omega, v \in \mathcal{C}^\infty$ are equal to 1 on $\operatorname{supp} f$, $\operatorname{supp} g$ and to 0 outside of neighborhoods of $\operatorname{supp} f$, $\operatorname{supp} g$, respectively (see, e.g., [1]).

It is known [1] that $\mathcal{S}$ is a nuclear (F) space, and $\mathcal{S}'$ is a nuclear (DF) space. From the duality theory as well as from theory of nuclear spaces [11] it follows that $\mathcal{S}_+$ is a nuclear (F) space, and $\mathcal{S}'_+$ is a nuclear (DF) space. Note that the space $\mathcal{S}_+$ is continuously and densely embedded into $\mathcal{S}'_+$ (it follows from the embedding $\mathcal{S} \hookrightarrow \mathcal{S}'$).

Polynomial tempered distributions. To define the locally convex space $\mathrm{P}(^n\mathcal{S}_+)$ of n-homogeneous polynomials on $\mathcal{S}_+$ we use the canonical topological linear isomorphism $\mathrm{P}(^n\mathcal{S}_+) \simeq (\otimes_{s,\mathrm{p}}^n \mathcal{S}_+)'$ described in [5]. Namely, given a functional $p_n \in (\otimes_{s,\mathrm{p}}^n \mathcal{S}_+)'$, we define an n-homogeneous polynomial $P_n \in \mathrm{P}(^n\mathcal{S}_+)$ by

$$P_n(\varphi) := \langle p_n, \varphi^{\otimes n} \rangle, \qquad \varphi \in \mathcal{S}_+.$$

We equip the space $\mathrm{P}(^n\mathcal{S}_+)$ with the topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{S}_+$. Set $\mathrm{P}(^0\mathcal{S}_+) := \mathbb{C}$. The space $\mathrm{P}(\mathcal{S}_+)$ of continuous polynomials on $\mathcal{S}_+$ is defined as the complex linear span of all $\mathrm{P}(^n\mathcal{S}_+)$ endowed with the topology $\mathfrak{b}$. We will denote by $\mathrm{P}'(\mathcal{S}_+)$, $\mathrm{P}'(^n\mathcal{S}_+)$ the strong duals. Similar spaces $\mathrm{P}(\mathcal{S}'_+)$, $\mathrm{P}(^n\mathcal{S}'_+)$ and their duals $\mathrm{P}'(\mathcal{S}'_+)$, $\mathrm{P}'(^n\mathcal{S}'_+)$ we define for $\mathcal{S}'_+$. Elements of the space $\mathrm{P}'(\mathcal{S}'_+)$ we will call the *polynomial tempered distributions.*

In the sequel, let $\times_{n\in\mathbb{Z}_+}(\otimes_{s,\mathrm{p}}^n \mathcal{S}'_+)$ denotes the Cartesian product and $\bigoplus_{n\in\mathbb{Z}_+}(\otimes_{s,\mathrm{p}}^n \mathcal{S}_+)$ denotes the direct sum of the symmetric tensor powers of corresponding spaces. Note that elements of the direct sum consist of finite but not fixed number of addends. To simplify notations, we denote $\Gamma(\mathcal{S}_+) := \bigoplus_{n\in\mathbb{Z}_+}(\otimes_{s,\mathrm{p}}^n \mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+) := \times_{n\in\mathbb{Z}_+}(\otimes_{s,\mathrm{p}}^n \mathcal{S}'_+)$. Elements of the spaces $\Gamma(\mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+)$ we write in the form $\mathbf{p} = \bigoplus_{n\in\mathbb{Z}_+} p_n = (p_0, p_1, \ldots, p_m, 0, \ldots)$ for some $m \in \mathbb{N}$ and $\mathbf{f} = \times_{n\in\mathbb{Z}_+} f_n = (f_0, f_1, \ldots, f_n, \ldots)$, respectively.

From [9, Proposition 2.1] it follows the next assertion.

Proposition 1. *For all* $n \in \mathbb{N}$, *there exist the linear topological isomorphisms* Υ_n, Ψ_n *and their linear extensions* Υ, Ψ *such that*

$$\Upsilon_n : \mathrm{P}({}^n\mathcal{S}'_+) \to \otimes^n_{s,\mathrm{p}} \mathcal{S}_+, \qquad \Upsilon : \mathrm{P}(\mathcal{S}'_+) \to \Gamma(\mathcal{S}_+),$$

$$\Psi_n : \mathrm{P}({}^n\mathcal{S}_+) \to \otimes^n_{s,\mathrm{p}} \mathcal{S}'_+, \qquad \Psi : \mathrm{P}'(\mathcal{S}'_+) \to \Gamma(\mathcal{S}'_+).$$

Annihilation and creation operators. In quantum field theory (see, e.g., [10]) an element

$$\phi_f = \times_{n\in\mathbb{Z}_+} \frac{f^{\otimes n}}{n!} = \left(1, f, \frac{f^{\otimes 2}}{2!}, \ldots, \frac{f^{\otimes n}}{n!}, \ldots\right) \in \Gamma(\mathcal{S}'_+), \qquad f \in \mathcal{S}'_+,$$

is usually referred as an exponential (or coherent) vector. In particular, $\phi_0 = (1, 0, 0, \ldots)$ is called a vacuum vector. Let us define a vector

$$\phi_{\varphi,m} = \left(1, \varphi, \frac{\varphi^{\otimes 2}}{2!}, \ldots, \frac{\varphi^{\otimes m}}{m!}, 0, \ldots\right) \in \Gamma(\mathcal{S}_+), \qquad \varphi \in \mathcal{S}_+, \qquad m \in \mathbb{N}.$$

Let by definition $\phi_{\varphi,0}$ be the vacuum vector. Note that the set $\{\phi_{\varphi,m} : \varphi \in \mathcal{S}_+, m \in \mathbb{Z}\}$ spans a dense subspace of $\Gamma(\mathcal{S}_+)$.

The annihilation operator at a point $t \in \mathbb{R}_+$, denoted by a_t, is a unique operator in $\mathcal{L}(\Gamma(\mathcal{S}_+))$ having the property

$$a_t \phi_{\varphi,m} = \varphi(t)\phi_{\varphi,m-1}, \qquad \varphi \in \mathcal{S}_+, \qquad m \in \mathbb{N}. \tag{1}$$

A creation operator $a'_t \in \mathcal{L}(\Gamma(\mathcal{S}'_+))$ at a point $t \in \mathbb{R}_+$ is, by definition, the adjoint operator to annihilation one with respect to duality $\langle \Gamma(\mathcal{S}'_+), \Gamma(\mathcal{S}_+)\rangle$. It may be defined as follows

$$a'_t : (1, f, f^{\otimes 2}, \ldots, f^{\otimes n}, \ldots) \mapsto (0, \delta_t, 2\delta_t \otimes_s f, \ldots, (n+1)\delta_t \otimes_s f^{\otimes n}, \ldots). \tag{2}$$

2. Translation operator and Gâteaux derivative. Let $P \in \mathrm{P}(\mathcal{S}'_+)$ be a continuous polynomial. Let us define the translation operator by

$$T_g P(f) = P(f+g), \qquad f \in \mathcal{S}'_+,$$

for any $g \in \mathcal{S}'_+$. It is easy to see, that $T_g \in \mathcal{L}(\mathrm{P}(\mathcal{S}'_+))$ for all $g \in \mathcal{S}'_+$.

Since the map Υ from Proposition 1 is topological isomorphism, the following operator $\mathbf{T}_g := \Upsilon \circ T_g \circ \Upsilon^{-1}$ is well defined on the space $\Gamma(\mathcal{S}_+)$, i.e., $\mathbf{T}_g \in \mathcal{L}(\Gamma(\mathcal{S}_+))$.

For any $\varphi \in \mathcal{S}_+$, denote $\boldsymbol{\varphi}_m := (1, \varphi, \varphi^{\otimes 2}, \ldots, \varphi^{\otimes m}, 0, \ldots) \in \Gamma(\mathcal{S}_+)$, $m \in \mathbb{N}$. The operator $\mathbf{T}_g$ acts on the element $\boldsymbol{\varphi}_m$ as follows

$$\mathbf{T}_g \boldsymbol{\varphi}_m = \left(\left(\sum_{i=k}^{m} \frac{i!}{k!(i-k)!} g^{\otimes(i-k)} \circledcirc_{i-k} \varphi^{\otimes i}\right)_{k=0}^{m}, 0, \ldots\right). \tag{3}$$

Here and below, the symbol $\circledcirc_r$ denotes the (right) r-contraction [6] of symmetric tensor product, i.e., $g^{\otimes r} \circledcirc_r \varphi^{\otimes s} := \langle g, \varphi\rangle^r \varphi^{\otimes(s-r)}$, $r \le s$.

Let us prove the equality (3). Let $P_{\varphi,m}$ be the polynomial that corresponds to the element $\boldsymbol{\varphi}_m$, i.e., $P_{\varphi,m} = \sum_{k=0}^{m} \langle \cdot^{\otimes k}, \varphi^{\otimes k} \rangle = \sum_{k=0}^{m} \langle \cdot, \varphi\rangle^k$. Then by direct calculations we obtain

$$T_g P_{\varphi,m}(f) = P_{\varphi,m}(f+g) = 1 + \langle f+g,\varphi\rangle + \langle f+g,\varphi\rangle^2 + \ldots + \langle f+g,\varphi\rangle^m =$$
$$= 1 + \langle f,\varphi\rangle + \langle g,\varphi\rangle + \langle f,\varphi\rangle^2 + 2\langle f,\varphi\rangle\langle g,\varphi\rangle + \langle g,\varphi\rangle^2 + \ldots +$$
$$+ \langle f,\varphi\rangle^m + m\langle f,\varphi\rangle^{m-1}\langle g,\varphi\rangle + \ldots + \langle g,\varphi\rangle^m =$$
$$= 1 + \langle g,\varphi\rangle + \langle g,\varphi\rangle^2 + \ldots + \langle g,\varphi\rangle^m +$$
$$+ \langle f,\varphi\rangle + 2\langle g,\varphi\rangle\langle f,\varphi\rangle + \ldots + m\langle g,\varphi\rangle^{m-1}\langle f,\varphi\rangle + \ldots +$$
$$+ \langle f,\varphi\rangle^k + \ldots + \frac{m!}{k!(m-k)!}\langle g,\varphi\rangle^{m-k}\langle f,\varphi\rangle^k +$$
$$+ \ldots + \langle f,\varphi\rangle^m = \sum_{k=0}^{m}\left(\sum_{i=k}^{m}\frac{i!}{k!(i-k)!}\langle g,\varphi\rangle^{i-k}\right)\langle f,\varphi\rangle^k, \qquad (4)$$

which is eqiuivalent to (3).

A polynomial $P \in \mathrm{P}(\mathcal{S}'_+)$ (element $\mathbf{p} \in \Gamma(\mathcal{S}_+)$, respectively) is said to be *Gâteaux differentiable* if for any $g \in \mathcal{S}'_+$ the translation $T_{\varepsilon g}P$ ($\mathbf{T}_{\varepsilon g}\mathbf{p}$, respectively) is defined for small ε with $|\varepsilon| < \varepsilon_0$ and if

$$D_g P := \lim_{\varepsilon\to 0}\frac{T_{\varepsilon g}P - P}{\varepsilon} \qquad \left(\mathbf{D}_g\mathbf{p} := \lim_{\varepsilon\to 0}\frac{\mathbf{T}_{\varepsilon g}\mathbf{p} - \mathbf{p}}{\varepsilon}, \text{ respectively}\right)$$

converges in $\mathrm{P}(\mathcal{S}'_+)$ (in $\Gamma(\mathcal{S}_+)$, respectively) with respect to the topology of uniform convergence on bounded sets (direct sum topology, respectively). In this case D_g ($\mathbf{D}_g$, respectively) is called the Gâteaux derivative of the polynomial P (of the element $\mathbf{p}$, respectively) in the direction g. Note that Gâteaux derivative sometimes is called the Gross derivative [6] or the Hida derivative [8].

3. Gâteaux differentiability of polynomial test functions. For any function $\varphi \in \mathcal{S}_+$ let us denote the polynomial

$$P_{\varphi,m} = \sum_{k=0}^{m}\left\langle \cdot^{\otimes k}, \varphi^{\otimes k}\right\rangle = \sum_{k=0}^{m}\langle \cdot,\varphi\rangle^k$$

belonging to $\mathrm{P}(\mathcal{S}'_+)$.

Theorem 1. *Every polynomial from the space* $\mathrm{P}(\mathcal{S}'_+)$ *is Gâteaux differentiable and*

$$D_g P_{\varphi,m} = \sum_{k=0}^{m-1}(k+1)\langle g,\varphi\rangle\left\langle \cdot^{\otimes k},\varphi^{\otimes k}\right\rangle = \sum_{k=0}^{m-1}(k+1)\langle g,\varphi\rangle\langle \cdot,\varphi\rangle^k. \qquad (5)$$

Moreover, Gâteaux derivative D_g is a linear continuous operator on the space $\mathrm{P}(\mathcal{S}'_+)$.

P r o o f. Using (4) we can write

$$\frac{T_{\varepsilon g}P_{\varphi,m}(f) - P_{\varphi,m}(f)}{\varepsilon} =$$
$$= \frac{\sum_{k=0}^{m}\left(\sum_{i=k}^{m}\frac{i!}{k!(i-k)!}\varepsilon^{i-k}\langle g,\varphi\rangle^{i-k}\right)\langle f,\varphi\rangle^k - \sum_{k=0}^{m}\langle f,\varphi\rangle^k}{\varepsilon} =$$

$$= \sum_{k=0}^{m} \frac{\left(\sum_{i=k}^{m} \frac{i!}{k!(i-k)!} \varepsilon^{i-k} \langle g, \varphi \rangle^{i-k}\right) \langle f, \varphi \rangle^{k} - \langle f, \varphi \rangle^{k}}{\varepsilon} =$$

$$= \sum_{k=0}^{m-1} \frac{\sum_{i=k+1}^{m} \frac{i!}{k!(i-k)!} \varepsilon^{i-k} \langle g, \varphi \rangle^{i-k} \langle f, \varphi \rangle^{k}}{\varepsilon} =$$

$$= \sum_{k=0}^{m-1} (k+1) \langle g, \varphi \rangle \langle f, \varphi \rangle^{k} +$$

$$+ \sum_{k=0}^{m-2} \sum_{i=k+2}^{m} \frac{i!}{k!(i-k)!} \varepsilon^{i-k-1} \langle g, \varphi \rangle^{i-k} \langle f, \varphi \rangle^{k} .$$

It is clear that the second sum tends to zero as $\varepsilon \to 0$, because of $i - k - 1 \geq 1$ in the sum.

Clearly, Gâteaux derivative is a linear operator. Show its continuity. Let B be any bounded set in $\mathcal{S}'_{+}$. Then for the polynomial $P_{\varphi,m} = \sum_{k=0}^{m} \langle \cdot, \varphi \rangle^{k}$ with $\varphi \in \mathcal{S}_{+}$ we have

$$\sup_{f \in B} \left| D_g P_{\varphi,m}(f) \right| = \sup_{f \in B} \left| \sum_{k=0}^{m-1} (k+1) \langle g, \varphi \rangle \langle f, \varphi \rangle^{k} \right| \leq$$

$$\leq \sup_{h \in B_1} \left| \sum_{k=0}^{m-1} (k+1) \langle h, \varphi \rangle \langle h, \varphi \rangle^{k} \right| \leq$$

$$\leq m \sup_{h \in B_1} \left| \sum_{k=0}^{m-1} \langle h, \varphi \rangle^{k+1} \right| \leq m \sup_{h \in B_1} \left| \sum_{k=0}^{m} \langle h, \varphi \rangle^{k} \right| =$$

$$= m \sup_{h \in B_1} \left| P_{\varphi,m}(h) \right|,$$

where B_1 is any bounded set, containing B and the element $g \in \mathcal{S}'_{+}$. ♦

Corollary 1. *Every element from the space* $\Gamma(\mathcal{S}_{+})$ *is Gâteaux differentiable and*

$$\mathbf{D}_{gm} \boldsymbol{\varphi}_m = \left((kg \circledcirc_1 \varphi^{\otimes k})_{k=1}^{m}, 0, \ldots \right) =$$

$$= \left(\langle g, \varphi \rangle, 2 \langle g, \varphi \rangle \varphi, \ldots, m \langle g, \varphi \rangle \varphi^{\otimes (m-1)}, 0, \ldots \right) \qquad (6)$$

for all $\boldsymbol{\varphi}_m = (1, \varphi, \varphi^{\otimes 2}, \ldots, \varphi^{\otimes m}, 0, \ldots) \in \Gamma(\mathcal{S}_{+})$, with $\varphi \in \mathcal{S}_{+}$. Moreover, Gâteaux derivative $\mathbf{D}_g$ is a linear continuous operator on the space $\Gamma(\mathcal{S}_{+})$.

P r o o f. The formula (6) is a consequence of (5). Linearity of $\mathbf{D}_g$ is obvious.

Let us show the continuity. Let $\{\mathbf{p}_k\}_{k \in \mathbb{N}}$ be a sequence tending to zero as $k \to \infty$ in the topology of $\Gamma(\mathcal{S}_{+})$, where $\mathbf{p}_k = \bigoplus_{n \in \mathbb{Z}_{+}} p_{n,k}$ with $p_{n,k} \in \otimes_{s,\mathrm{p}}^{n} \mathcal{S}_{+}$ for all $k \in \mathbb{N}$. Without loss of generality we can assume that elements $p_{n,k}$ have the form $p_{n,k} = \varphi_k^{\otimes n}$ with $\varphi_k \in \mathcal{S}_{+}$, $k \in \mathbb{N}$. From definition of direct sum topology it follows that $\varphi_k \to 0$ as $k \to \infty$ in the space $\mathcal{S}_{+}$. The sequence $\langle g, \varphi_k \rangle$ tends to zero because of continuity of the distribution $g \in \mathcal{S}'_{+}$. It follows that $\mathbf{D}_g \mathbf{p}_k = \left(\langle g, \varphi_k \rangle, 2 \langle g, \varphi_k \rangle \varphi_k, \ldots, m \langle g, \varphi_k \rangle \varphi_k^{\otimes (m-1)}, 0, \ldots \right)$ tends to zero as well. ♦

Corollary 2. *Every element* $\phi_{\varphi,m} \in \Gamma(\mathcal{S}_+)$ *is Gâteaux differentiable and*

$$\mathbf{D}_g \phi_{\varphi,m} = \langle g, \varphi \rangle \phi_{\varphi,m-1}.$$

Corollary 3. *For* $g = \delta_t$, *we obtain* $\mathbf{D}_{\delta_t} \phi_{\varphi,m} = \varphi(t)\phi_{\varphi,m-1}$, *hence* $\mathbf{D}_{\delta_t} = a_t$ *is the annihilation operator* (1) *at a point* $t \in \mathbb{R}_+$.

In [2] it is shown, that the space $\Gamma(\mathcal{S}_+)$ is an algebra with respect to the Wick product

$$\mathbf{p} \lozenge \mathbf{q} := \bigoplus_{n\in\mathbb{Z}_+} \sum_{k=0}^{n} p_k \otimes_s q_{n-k}, \qquad \mathbf{p} = \bigoplus_{n\in\mathbb{Z}_+} p_n, \qquad \mathbf{q} = \bigoplus_{n\in\mathbb{Z}_+} q_n.$$

Theorem 2. *For each* $g \in \mathcal{S}'_+$, *Gâteaux derivative* $\mathbf{D}_g$ *is a continuous differentiation of the algebra* $\{\Gamma(\mathcal{S}_+), \lozenge\}$, *i.e.,*

$$\mathbf{D}_g(\mathbf{p} \lozenge \mathbf{q}) = \mathbf{D}_g\mathbf{p} \lozenge \mathbf{q} + \mathbf{p} \lozenge \mathbf{D}_g\mathbf{q}, \qquad \mathbf{p}, \mathbf{q} \in \Gamma(\mathcal{S}_+).$$

P r o o f. Let $\mathbf{p} = \bigoplus_{n\in\mathbb{Z}_+} \varphi^{\otimes n}$, $\mathbf{q} = \bigoplus_{n\in\mathbb{Z}_+} \psi^{\otimes n}$ with $\varphi, \psi \in \mathcal{S}_+$. Directly from the definitions we obtain

$$\mathbf{D}_g\mathbf{p} \lozenge \mathbf{q} = \Big(\bigoplus_{n\in\mathbb{N}} n \langle g, \varphi \rangle \varphi^{\otimes(n-1)} \Big) \lozenge \Big(1 \oplus \bigoplus_{n\in\mathbb{N}} \psi^{\otimes n} \Big) =$$
$$= \bigoplus_{n\in\mathbb{N}} \sum_{k=1}^{n} k \langle g, \varphi \rangle (\varphi^{\otimes(k-1)} \otimes_s \psi^{\otimes(n-k)}).$$

Analogously,

$$\mathbf{p} \lozenge \mathbf{D}_g\mathbf{q} = \bigoplus_{n\in\mathbb{N}} \sum_{k=1}^{n} k \langle g, \psi \rangle (\psi^{\otimes(k-1)} \otimes_s \varphi^{\otimes(n-k)}).$$

Therefore,

$$\mathbf{D}_g\mathbf{p} \lozenge \mathbf{q} + \mathbf{p} \lozenge \mathbf{D}_g\mathbf{q} = \bigoplus_{n\in\mathbb{N}} \sum_{k=1}^{n} \big[k \langle g, \varphi \rangle (\varphi^{\otimes(k-1)} \otimes_s \psi^{\otimes(n-k)}) +$$
$$+ k \langle g, \psi \rangle (\psi^{\otimes(k-1)} \otimes_s \varphi^{\otimes(n-k)}) \big] = \bigoplus_{n\in\mathbb{N}} \big[n \langle g, \psi \rangle \psi^{\otimes(n-1)} +$$
$$+ (\langle g, \varphi \rangle \psi^{\otimes(n-1)} + (n-1) \langle g, \psi \rangle \varphi \otimes_s \psi^{\otimes(n-2)}) + \ldots +$$
$$+ (\langle g, \psi \rangle \varphi^{\otimes(n-1)} + (n-1) \langle g, \varphi \rangle \psi \otimes_s \varphi^{\otimes(n-2)}) +$$
$$+ n \langle g, \varphi \rangle \varphi^{\otimes(n-1)} \big] = \bigoplus_{n\in\mathbb{N}} \big[ng \circledcirc_1 \psi^{\otimes n} + ng \circledcirc_1 (\varphi \otimes_s \psi^{\otimes(n-1)}) +$$
$$+ \ldots + ng \circledcirc_1 (\psi \otimes_s \varphi^{\otimes(n-1)}) + ng \circledcirc_1 \varphi^{\otimes n} \big] =$$
$$= \bigoplus_{n\in\mathbb{N}} ng \circledcirc_1 \Big(\sum_{k=0}^{n} (\varphi^{\otimes k} \otimes_s \psi^{\otimes(n-k)}) \Big) = \mathbf{D}_g(\mathbf{p} \lozenge \mathbf{q}).$$

The space $\mathrm{P}(\mathcal{S}'_+)$ is a topological algebra with the scalar unit and the pointwise multiplication

$$P(f) \cdot Q(f) = \sum_{n\in\mathbb{Z}_+}^{\text{fin}} \sum_{k=0}^{n} P_k(f) \cdot Q_{n-k}(f), \qquad f \in \mathcal{S}'_+.$$

Here and below, the symbol $\sum\limits_{n\in\mathbb{Z}_+}^{\text{fin}}$ means that in the sum there is a finite but not fixed number of addends.

It is easy to see that such product of polynomials corresponds to the Wick product of according elements with respect to isomorphism Υ, i.e.,

$$\Upsilon(P \cdot Q) = \mathbf{p} \lozenge \mathbf{q},$$

where $\Upsilon(P) = \mathbf{p}$, $\Upsilon(Q) = \mathbf{q}$.

As a consequence we get the following

Corollary 4. *For each* $g \in \mathcal{S}'_+$, *Gâteaux derivative* D_g *is a continuous differentiation of the algebra* $\{\mathrm{P}(\mathcal{S}'_+), \cdot\}$, *i.e.,*

$$D_g(P \cdot Q) = D_g P \cdot Q + P \cdot D_g Q, \qquad P, Q \in \mathrm{P}(\mathcal{S}'_+).$$

4. Gâteaux differentiability of polynomial distributions. For an arbitrary $g \in \mathcal{S}'_+$ let $\mathbf{D}'_g$ and D'_g denote adjoint operators to Gâteaux derivative with respect to the duality $\langle \Gamma(\mathcal{S}'_+), \Gamma(\mathcal{S}_+) \rangle$ and $\langle \mathrm{P}'(\mathcal{S}'_+), \mathrm{P}(\mathcal{S}'_+) \rangle$, respectively.

Theorem 3. *The operator* $\mathbf{D}'_g \in \mathcal{L}(\Gamma(\mathcal{S}'_+))$ *is a linear and continuous operator acting on any element* $\mathbf{f} = \times_{n \in \mathbb{Z}_+} f_n \in \Gamma(\mathcal{S}'_+)$ *as follows*

$$\mathbf{D}'_g \mathbf{f} = \mathbf{D}'_g \big[\times_{n \in \mathbb{Z}_+} f_n \big] = \big(0, f_0 g, 2g \otimes_s f_1, 3g \otimes_s f_2, \ldots, (n+1) g \otimes_s f_n, \ldots \big).$$

P r o o f. Corollary 1 implies the linearity and continuity of the operator $\mathbf{D}'_g$. From definition of Gâteaux derivative as well as from (6) by direct calculations we obtain

$$\begin{aligned}
\Big\langle \mathbf{D}'_g [\times_{n \in \mathbb{Z}_+} f_n], \mathbf{p} \Big\rangle &= \Big\langle \times_{n \in \mathbb{Z}_+} f_n, \mathbf{D}_g \big[\textstyle\bigoplus_{n \in \mathbb{Z}_+} p_n \big] \Big\rangle = \\
&= \Big\langle \times_{n \in \mathbb{Z}_+} f_n, \textstyle\bigoplus_{n \in \mathbb{N}} n g \circledcirc_1 p_n \Big\rangle = \\
&= \big\langle (f_0, f_1, f_2, \ldots), (g \circledcirc_1 p_1, 2g \circledcirc_1 p_2, 3g \circledcirc_1 p_3, \ldots) \big\rangle = \\
&= 0 + \sum_{n \in \mathbb{N}}^{\mathrm{fin}} n \langle g \otimes_s f_{n-1}, p_n \rangle = \\
&= \Big\langle 0 \times \times_{n \in \mathbb{N}} n g \otimes_s f_{n-1}, \textstyle\bigoplus_{n \in \mathbb{Z}_+} p_n \Big\rangle = \\
&= \big\langle (0, f_0 g, 2g \otimes_s f_1, 3g \otimes_s f_2, \ldots, (n+1) g \otimes_s f_n, \ldots), \mathbf{p} \big\rangle
\end{aligned}$$

for any $\mathbf{p} = \bigoplus_{n \in \mathbb{Z}_+} p_n \in \Gamma(\mathcal{S}_+)$. ♦

As a consequence for $g = \delta_t$ we obtain the following assertion (compare with (2)).

Corollary 5. *The operator* $\mathbf{D}'_{\delta_t}$ *is the creation operator* a'_t *at a point* $t \in \mathbb{R}_+$

$$\mathbf{D}'_{\delta_t} \big[\times_{n \in \mathbb{Z}_+} f^{\otimes n} \big] = \big(0, \delta_t, 2\delta_t \otimes_s f, 3\delta_t \otimes_s f^{\otimes 2}, \ldots, (n+1)\delta_t \otimes_s f^{\otimes n}, \ldots \big).$$

Corollary 6. *The operator* $D'_g \in \mathcal{L}(\mathrm{P}(\mathcal{S}'_+))$ *is a linear and continuous operator.*

In [2] it is shown, that the space $\Gamma(\mathcal{S}'_+)$ is an algebra with respect to the Wick product

$$\mathbf{f} \lozenge \mathbf{h} := \mathop{\times}_{n \in \mathbb{Z}_+} \sum_{k=0}^{n} f_k \otimes_s h_{n-k}, \qquad \mathbf{f} = \mathop{\times}_{n \in \mathbb{Z}_+} f_n, \qquad \mathbf{h} = \mathop{\times}_{n \in \mathbb{Z}_+} h_n.$$

Theorem 4. *For each* $g \in \mathcal{S}'_+$ *generalized Gâteaux derivative* $\mathbf{D}'_g$ *is a continuous differentiation of the algebra* $\{\Gamma(\mathcal{S}'_+), \lozenge\}$, *i.e.,*

$$\mathbf{D}'_g(\mathbf{f} \lozenge \mathbf{h}) = \mathbf{D}'_g \mathbf{f} \lozenge \mathbf{h} + \mathbf{f} \lozenge \mathbf{D}'_g \mathbf{h}, \qquad \mathbf{f}, \mathbf{h} \in \Gamma(\mathcal{S}'_+).$$

P r o o f. Let $\mathbf{f} = \times_{n \in \mathbb{Z}_+} f^{\otimes n}$, $\mathbf{h} = \times_{n \in \mathbb{Z}_+} h^{\otimes n}$ with $f, h \in \mathcal{S}'_+$. Analogously as in the proof of the Theorem 2 directly from the definitions we obtain the formulas

$$\mathbf{D}'_g \mathbf{f} \lozenge \mathbf{h} = 0 \times \bigtimes_{n\in\mathbb{N}} \sum_{k=1}^{n} kg \otimes_s f^{\otimes(k-1)} \otimes_s h^{\otimes(n-k)}$$

and

$$\mathbf{f} \lozenge \mathbf{D}'_g \mathbf{h} = 0 \times \bigtimes_{n\in\mathbb{N}} \sum_{k=1}^{n} kg \otimes_s h^{\otimes(k-1)} \otimes_s f^{\otimes(n-k)} .$$

Therefore,

$$\mathbf{D}'_g \mathbf{f} \lozenge \mathbf{h} + \mathbf{f} \lozenge \mathbf{D}'_g \mathbf{h} =$$

$$= 0 \times \bigtimes_{n\in\mathbb{N}} (n+1) g \otimes_s \sum_{k=1}^{n} f^{\otimes(k-1)} \otimes_s h^{\otimes(n-k)} = \mathbf{D}'_g (\mathbf{f} \lozenge \mathbf{h}) .$$

The multiplication in $\mathrm{P}(\mathcal{S}'_+)$ can be uniquely extended to the multiplication in $\mathrm{P}'(\mathcal{S}'_+)$, so $\{\mathrm{P}'(\mathcal{S}'_+), \cdot\}$ is an algebra (see [2, 9]).

Corollary 7. *For each* $g \in \mathcal{S}'_+$ *generalized Gâteaux derivative* D'_g *is a continuous differentiation of the algebra* $\{\mathrm{P}'(\mathcal{S}'_+), \cdot\}$.

5. Connection with differentiations on a Fock type space. Let the operator $\eth \in \Gamma(\mathcal{S}_+)$ be given by $\eth\boldsymbol{\varphi} := \bigoplus_{n\in\mathbb{Z}_+} D^{\{\otimes\}_n}[\varphi^{\otimes n}]$ for any $\boldsymbol{\varphi} = \bigoplus_{n\in\mathbb{Z}_+} \varphi^{\otimes n} \in \Gamma(\mathcal{S}_+)$, $\varphi \in \mathcal{S}_+$, where the operators $D^{\{\otimes\}_n}$ are defined by the formulas $D^{\{\otimes\}_0}[\varphi^{\otimes 0}] := 0$ and $D^{\{\otimes\}_n}[\varphi^{\otimes n}] := \sum_{j=1}^{n} \underbrace{\varphi \otimes \ldots \otimes \varphi \otimes \varphi'}_{j} \otimes \underbrace{\varphi \otimes \ldots \otimes \varphi}_{n-j}$ for $n \in \mathbb{N}$.

In [2, 9] it is proved that such operator generate a strong continuous semigroup of polynomial shifts on the Fock type space $\Gamma(\mathcal{S}_+)$. Moreover, the operator $\eth$ is the differentiation in the sense of Leibnitz property.

Let us modify the definition of the operator $\eth$ in the following way. For a fixed $t > 0$ let $\eth(t) \in \Gamma(\mathcal{S}_+)$ be defined by

$$\eth(t)\boldsymbol{\varphi} := \bigoplus_{n\in\mathbb{N}} D_t^{\{\otimes\}_n}[\varphi^{\otimes n}],$$

where $D_t^{\{\otimes\}_n}[\varphi^{\otimes n}] := \sum_{j=1}^{n} \underbrace{\varphi \otimes \ldots \otimes \varphi \otimes \varphi'(t)}_{j} \otimes \underbrace{\varphi \otimes \ldots \otimes \varphi}_{n-j}$, $n \in \mathbb{N}$. Since $\varphi'(t)$ is a constant, it is clear that

$$\eth(t)\left[\bigoplus_{n\in\mathbb{Z}_+} \varphi^{\otimes n}\right] = \bigoplus_{n\in\mathbb{N}} n\varphi'(t)\varphi^{\otimes(n-1)} . \tag{7}$$

Theorem 5. *For* $g = -\delta'_t$ *we obtain* $D_{-\delta'_t} = \eth(t)$ *with fixed* $t > 0$.

P r o o f. For any $\boldsymbol{\varphi}_m = (1, \varphi, \varphi^{\otimes 2}, \ldots, \varphi^{\otimes m}, 0, \ldots) \in \Gamma(\mathcal{S}_+)$ with $\varphi \in \mathcal{S}_+$ and fixed $t \in \mathbb{R}_+$ we have

$$\begin{aligned}\mathbf{D}_{-\delta'_t} \boldsymbol{\varphi}_m &= (\langle -\delta'_t, \varphi\rangle, 2\langle -\delta'_t, \varphi\rangle \varphi, 3\langle -\delta'_t, \varphi\rangle \varphi^{\otimes 2}, \ldots, \\ &\quad m\langle -\delta'_t, \varphi\rangle \varphi^{\otimes(m-1)}, 0, \ldots) = (\langle \delta_t, \varphi'\rangle, 2\langle \delta_t, \varphi'\rangle \varphi, \\ &\quad 3\langle \delta_t, \varphi'\rangle \varphi^{\otimes 2}, \ldots, m\langle \delta_t, \varphi'\rangle \varphi^{\otimes(m-1)}, 0, \ldots) = \\ &= \left(\varphi'(t), 2\varphi'(t)\varphi, 3\varphi'(t)\varphi^{\otimes 2}, \ldots, m\varphi'(t)\varphi^{\otimes(m-1)}, 0, \ldots\right),\end{aligned}$$

which coincides with (7). ♦

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ДИФЕРЕНЦІЙОВНІСТЬ ЗА ҐАТО ПОЛІНОМІАЛЬНИХ ОСНОВНИХ ТА УЗАГАЛЬНЕНИХ ФУНКЦІЙ

Нехай $\mathcal{S}_+$ *і* $\mathcal{S}'_+$ *– простори Шварца швидкоспадних функцій і розподілів повільного зростання на* $\mathbb{R}_+$ *відповідно. Нехай* $\mathrm{P}(\mathcal{S}'_+)$ *– простір неперервних поліномів над* $\mathcal{S}'_+$, *а* $\mathrm{P}'(\mathcal{S}'_+)$ *– простір, сильно спряжений до* $\mathrm{P}(\mathcal{S}'_+)$. *Ці простори можна подати у формі просторів типу Фока* $\Gamma(\mathcal{S}_+) := \bigoplus_{n\in\mathbb{Z}_+}(\otimes^n_{s,\mathfrak{p}}\mathcal{S}_+)$ *і* $\Gamma(\mathcal{S}'_+) := \times_{n\in\mathbb{Z}_+}(\otimes^n_{s,\mathfrak{p}}\mathcal{S}'_+)$ *відповідно. Досліджується диференційовність за Ґато елементів просторів* $\mathrm{P}(\mathcal{S}'_+)$, $\mathrm{P}'(\mathcal{S}'_+)$, $\Gamma(\mathcal{S}_+)$ *та* $\Gamma(\mathcal{S}'_+)$. *Розглянуто зв'язок похідної за Ґато з операторами народження і знищення на просторах типу Фока, а також із диференціюваннями на* $\Gamma(\mathcal{S}_+)$ *та* $\Gamma(\mathcal{S}'_+)$.

ДИФФЕРЕНЦИРУЕМОСТЬ ПО ГАТО ПОЛИНОМИАЛЬНЫХ ОСНОВНЫХ И ОБОБЩЕННЫХ ФУНКЦИЙ

Пусть $\mathcal{S}_+$ *и* $\mathcal{S}'_+$ *– пространства Шварца быстроубывающих функций и медленно растущих распределений на* $\mathbb{R}_+$ *соответственно. Пусть* $\mathrm{P}(\mathcal{S}'_+)$ *– пространство непрерывных полиномов над* $\mathcal{S}'_+$, *а* $\mathrm{P}'(\mathcal{S}'_+)$ *– пространство, сильно сопряженное к* $\mathrm{P}(\mathcal{S}'_+)$. *Эти пространства имеют представления в форме пространств типа Фока* $\Gamma(\mathcal{S}_+) := \bigoplus_{n\in\mathbb{Z}_+}(\otimes^n_{s,\mathfrak{p}}\mathcal{S}_+)$ *и* $\Gamma(\mathcal{S}'_+) := \times_{n\in\mathbb{Z}_+}(\otimes^n_{s,\mathfrak{p}}\mathcal{S}'_+)$ *соответственно. Исследуется дифференцируемость по Гато элементов пространств* $\mathrm{P}(\mathcal{S}'_+)$, $\mathrm{P}'(\mathcal{S}'_+)$, $\Gamma(\mathcal{S}_+)$ *и* $\Gamma(\mathcal{S}'_+)$. *Рассмотрена связь производной по Гато с операторами рождения и уничтожения на пространствах типа Фока, а также с дифференцированиями на* $\Gamma(\mathcal{S}_+)$ *и* $\Gamma(\mathcal{S}'_+)$.

Vasyl Stefanyk Precarpat.
Nat. Univ., Ivano-Frankivsk

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