UDK 539.3

Y. Z. Povstenko

FUNDAMENTAL SOLUTIONS TO ROBIN BOUNDARY-VALUE PROBLEMS FOR TIME-FRACTIONAL HEAT CONDUCTION EQUATION IN A HALF-LINE

The time-fractional heat conduction equation with the Caputo derivative of the order $0 < \alpha \le 2$ is considered in a half-line. Two types of Robin boundary condition are examined: the mathematical condition with the prescribed linear combination of the values of temperature and the values of its normal derivative and the physical condition with the prescribed linear combination of the values of temperature and the values of the heat flux at the boundary of the domain. These two types of Robin boundary condition coincide only in the case of classical heat conduction equation.

Introduction. The time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T, \qquad 0 < \alpha \le 2, \tag{1}$$

describes many important physical phenomena in different media [3, 5, 11, 16, 24].

Equation (1) can be obtained as a consequence of the balance equation for energy and the generalized Fourier law – the time-nonlocal dependence between the heat flux vector $\mathbf{q}$ and the temperature gradient ∇T with the long-tail power kernel [15, 16, 21, 22] (see also [8]):

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\nabla T(\tau)\,d\tau, \qquad 0 < \alpha \le 1, \tag{2}$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}\nabla T(\tau)\,d\tau, \qquad 1 < \alpha \le 2, \tag{3}$$

where k is the thermal conductivity, $\Gamma(\alpha)$ is the gamma function. Equations (2) and (3) can be interpreted in terms of fractional calculus:

$$\mathbf{q}(t) = -kD_{\mathrm{RL}}^{1-\alpha}\nabla T(t), \qquad 0 < \alpha \le 1, \tag{4}$$

$$\mathbf{q}(t) = -kI^{\alpha-1}\nabla T(t), \qquad 1 < \alpha \le 2. \tag{5}$$

Here $I^\alpha f(t)$ and $D_{\mathrm{RL}}^\alpha f(t)$ are the Riemann – Liouville fractional integral and derivative of the order α, respectively [2, 6, 10]:

$$I^n f(t) = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau, \qquad t > 0, \qquad \alpha > 0,$$

$$D_{\mathrm{RL}}^\alpha f(t) = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau\right],$$

$$t > 0, \quad n-1 < \alpha < n.$$

The constitutive equations (4) and (5) yield the time-fractional diffusion equation (1) with the Caputo fractional derivative

$$D_{\mathrm{C}}^\alpha f(t) \equiv \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}\,d\tau,$$

$$t > 0, \quad n-1 < \alpha < n.$$

If Eq. (1) is investigated in a bounded domain, the boundary conditions should be prescribed. The Dirichlet boundary condition specifies the value of temperature over the surface of the body under consideration

$$T\big|_S = g_0(\mathbf{x}_s, t) .$$

For time-fractional heat conduction equation (1), two types of Neumann boundary condition can be considered: the mathematical condition with the prescribed boundary value of the normal derivative of temperature

$$\left.\frac{\partial T}{\partial n}\right|_S = G_0(\mathbf{x}_s, t) ,$$

and the physical condition with the prescribed boundary value of the heat flux

$$D_{\mathrm{RL}}^{1-\alpha} \left.\frac{\partial T}{\partial n}\right|_S = G_0(\mathbf{x}_s, t), \qquad 0 < \alpha \le 1 ,$$

$$I^{\alpha-1} \left.\frac{\partial T}{\partial n}\right|_S = G_0(\mathbf{x}_s, t), \qquad 1 < \alpha \le 2 .$$

Similarly, the mathematical Robin boundary condition is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain

$$\left.\left(c_1 T + c_2 \frac{\partial T}{\partial n}\right)\right|_S = H_0(\mathbf{x}_s, t)$$

with some nonzero constants c_1 and c_2, while the physical Robin boundary condition specifies a linear combination of the values of temperature and the values of the heat flux at the boundary of the domain. For example, the condition of convective heat exchange between a body and the environment with the temperature T_e leads to

$$\left.\left(hT + k D_{RL}^{1-\alpha} \frac{\partial T}{\partial n}\right)\right|_S = hT_e(\mathbf{x}_s, t), \qquad 0 < \alpha \le 1 ,$$

$$\left.\left(hT + k I^{\alpha-1} \frac{\partial T}{\partial n}\right)\right|_S = hT_e(\mathbf{x}_s, t), \qquad 1 < \alpha \le 2 ,$$

where h is the convective heat transfer coefficient.

Starting from the pioneering papers [4, 12, 13, 23, 25], considerable interest have been shown in solutions to time-fractional heat conduction equation. Several problems have been solved for bounded domain with various kinds of boundary conditions [7, 9, 14, 17–20]. In the present paper, for the first time the fundamental solutions to time-fractional heat conduction equation in a half-line are studied under both the mathematical and physical Robin boundary conditions.

1. Mathematical Robin boundary condition. Consider the time-fractional heat conduction equation in a half-line

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < \infty, \qquad 0 < t < \infty, \qquad 0 < \alpha \le 2 , \tag{6}$$

under zero initial conditions

$$t = 0 : \qquad T = 0, \qquad 0 < \alpha \le 2 , \tag{7}$$

$$t = 0 : \qquad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2 , \tag{8}$$

and the mathematical Robin boundary condition

$$x = 0 : \qquad HT - \frac{\partial T}{\partial x} = q_0 \delta_+(t) . \tag{9}$$

The Laplace transform with respect to time t allows us to eliminate the time-differentiation in Eq. (6). Recall that the Caputo derivative for its Laplace transform requires the knowledge of the initial values of the function $f(0^+)$ and its integer derivatives $f^{(k)}(0^+)$ of order $k = 1,\ldots,n-1$ [6, 10]:

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \qquad n-1 < \alpha < n,$$

where s is the transform variable, the asterisk denotes the transform.

To solve Eq. (6) under the boundary condition (9) we will use the sin-cos-Fourier transform [1] with respect to the spatial coordinate x:

$$F\{f(x)\} = f^*(\xi) = \int_0^\infty K(x,\xi) f(x)\, dx,$$

$$F^{-1}\{f^*(\xi)\} = f(x) = \frac{2}{\pi}\int_0^\infty K(x,\xi) f^*(\xi)\, d\xi$$

with the kernel

$$K(x,\xi) = \frac{\xi\cos(x\xi) + H\sin(x\xi)}{\sqrt{\xi^2 + H^2}}.$$

Application of the sin-cos-Fourier transform to the second derivative of a function gives

$$F\left\{\frac{d^2 f(x)}{dx^2}\right\} = -\xi^2 f^*(\xi) - \frac{\xi}{\sqrt{\xi^2+H^2}}\left[\frac{df(x)}{dx} - Hf(x)\right]\Bigg|_{x=0}.$$

The solution of (6)–(9) in the transform domain has the form

$$T^{**}(\xi,s) = aq_0 \frac{\xi}{\sqrt{\xi^2+H^2}} \frac{1}{s^\alpha + a\xi^2},$$

and after inversion of the integral transforms we obtain

$$T(x,t) = \frac{2aq_0 t^{\alpha-1}}{\pi}\int_0^\infty \frac{\xi^2\cos(x\xi) + H\xi\sin(x\xi)}{\xi^2+H^2} E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\, d\xi, \tag{10}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function in two parameters

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha > 0, \qquad \beta > 0, \qquad z \in C.$$

Several particular cases of solution (10) can be considered. For the classical heat conduction equation $(\alpha = 1)$ we get

$$T(x,t) = aq_0\left[\frac{1}{\sqrt{\pi a t}}\exp\left(-\frac{x^2}{4at}\right) - \right.$$

$$\left. - H\exp(Hx + H^2 at)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{at}} + H\sqrt{at}\right)\right]. \tag{11}$$

In the case of the classical wave equation $(\alpha = 2)$ the solution is

$$T(x,t) = \begin{cases} \sqrt{a}\, q_0 e^{-H(\sqrt{a}t - x)}, & 0 < x < \sqrt{a}\,t, \\ 0, & \sqrt{a}\,t < x < \infty. \end{cases}$$

It is obvious that Eq. (10) for $H = 0$,

$$T(x,t) = \frac{2aq_0 t^{\alpha-1}}{\pi} \int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\cos(x\xi)\, d\xi\,, \tag{12}$$

represents the corresponding fundamental solution under the mathematical Neumann boundary condition.

2. Physical Robin boundary condition. Now we consider the time-fractional heat conduction equation in a half-line (6) under zero initial conditions (7) and (8) and the physical Robin boundary condition

$$x = 0: \qquad HT - D_{\mathrm{RL}}^{1-\alpha} \frac{\partial T}{\partial x} = q_0 \delta_+(t), \qquad 0 < \alpha \le 1\,,$$

$$x = 0: \qquad HT - I^{\alpha-1} \frac{\partial T}{\partial x} = q_0 \delta_+(t), \qquad 1 < \alpha \le 2\,.$$

The Laplace transform with respect to time t leads to the following equation

$$s^\alpha T^* = a \frac{\partial^2 T_*}{\partial x^2}, \qquad 0 < x < \infty, \qquad 0 < \alpha \le 2\,,$$

and the boundary condition

$$x = 0: \qquad s^{\alpha-1} H T^* - \frac{\partial T^*}{\partial x} = q_0 s^{\alpha-1}, \qquad 0 < \alpha \le 2\,.$$

In this case the kernel of the sin-cos-Fourier transform with respect to the spatial coordinate x has the more complicated form

$$K(x,\xi) = \frac{\xi\cos(x\xi) + s^{\alpha-1} H \sin(x\xi)}{\sqrt{\xi^2 + (s^{\alpha-1}H)^2}}\,,$$

and in the transform domain we get

$$T^{**}(\xi,s) = aq_0 \frac{\xi}{\sqrt{\xi^2 + (s^{\alpha-1}H)^2}} \frac{s^{\alpha-1}}{s^\alpha + a\xi^2}\,. \tag{13}$$

Inversion of the Laplace transform in Eq. (13) depends on the value of α. For $0 < \alpha \le 1$ we have

$$T(x,t) = \frac{2aq_0}{\pi} \int_0^\infty E_\alpha(-a\xi^2 t^\alpha)\cos(x\xi)\, d\xi - \frac{2aq_0}{\pi} \int_0^\infty \frac{H^2}{\xi^2}\cos(x\xi)\, d\xi \times$$

$$\times \int_0^t (t-\tau)^{1-2\alpha} E_{2-2\alpha,2-2\alpha}\left[-\frac{H^2}{\xi^2}(t-\tau)^{2-2\alpha}\right] E_\alpha(-a\xi^2\tau^\alpha)\, d\tau +$$

$$+ \frac{2aq_0}{\pi} \int_0^\infty \frac{H}{\xi}\sin(x\xi)\, d\xi \int_0^t (t-\tau)^{1-2\alpha}\tau^{\alpha-1} E_{2-2\alpha,2-2\alpha} \times$$

$$\times \left[-\frac{H^2}{\xi^2}(t-\tau)^{2-2\alpha}\right] E_{\alpha,\alpha}(-a\xi^2\tau^\alpha)\, d\tau\,, \tag{14}$$

whereas for $1 < \alpha \le 2$ we get

$$T(x,t) = \frac{2aq_0}{\pi} \int_0^\infty \frac{\xi}{H} t^{\alpha-1} E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\sin(x\xi)\, d\xi + \frac{2aq_0}{\pi} \int_0^\infty \frac{\xi^2}{H^2}\cos(x\xi)\, d\xi \times$$

$$\times \int_0^t (t-\tau)^{2\alpha-3} E_{2\alpha-2,2\alpha-2}\left[-\frac{\xi^2}{H^2}(t-\tau)^{2\alpha-2}\right] E_\alpha(-a\xi^2\tau^\alpha)\, d\tau -$$

$$-\frac{2aq_0}{\pi}\int_0^\infty \frac{\xi^3}{H^3}\sin(x\xi)\,d\xi\int_0^t (t-\tau)^{2\alpha-3}\tau^{\alpha-1}E_{2\alpha-2,2\alpha-2}\times$$

$$\times\left[-\frac{\xi^2}{H^2}(t-\tau)^{2\alpha-2}\right]E_{\alpha,\alpha}(-a\xi^2\tau^\alpha)\,d\tau\,. \qquad (15)$$

For the standard heat conduction equation $(\alpha = 1)$, we arrive at the solution (11). For the classical wave equation $(\alpha = 2)$, we obtain

$$T(x,t) = \frac{aq_0}{1+\sqrt{a}\,H}\delta(x-\sqrt{a}t)\,.$$

From Eq. (13) in the case $H = 0$ it follows that the fundamental solution to time-fractional heat conduction equation (6) under physical Neumann boudary condition is expressed as

$$T(x,t) = \frac{2aq_0}{\pi}\int_0^\infty E_\alpha(-a\xi^2t^\alpha)\cos(x\xi)\,d\xi\,. \qquad (16)$$

In the case of the classical heat conduction equation $(\alpha = 1)$, the solutions (10) and (14) or (15) of Eq. (6) under the mathematical and physical Robin boundary conditions coincide as well as the solutions (12) and (16) under the mathematical and physical Neumann boundary conditions, but for time-fractional heat conduction equation $(\alpha \neq 1)$ they are essentially different.

1. *Галицын А. С., Жуковский А. Н.* Интегральные преобразования и специальные функции в задачах теплопроводности. – Киев: Наук. думка, 1976. – 282 с.
2. *Самко С. Г., Килбас А. А., Маричев О. И.* Интегралы и производные дробного порядка и некоторые их приложения. – Минск: Наука и техника, 1987. – 688 с.
   Samko S. G., Kilbas A. A., Marichev O. I. Fractional integrals and derivatives: Theory and applications. – Newark: Gordon and Breach, 1993. – 587 p.
3. *Учайкин В. В.* Метод дробных производных. – Ульяновск: Артишок, 2008. – 510 с.
4. *Fujita Y.* Integrodifferential equation which interpolates the heat equation and the wave equation // Osaka J. Math. – 1990. – **27**, No. 2. – P. 309–321.
5. *Gafiychuk V. V., Datsko B. Yo.* Mathematical modeling of different types of instabilities in time fractional reaction-diffusion systems // Comput. Math. Appl. – 2010. – **59**, No. 3. – P. 1101–1107.
6. *Gorenflo R., Mainardi F.* Fractional calculus: integral and differential equations of fractional order // Fractals and Fractional Calculus in Continuum Mechanics / Eds. A. Carpinteri, F. Mainardi. – Wien: Springer, 1997. – P. 223–276.
7. *Gorenflo R., Mainardi F.* Signalling problem and Dirichlet–Neumann map for time-fractional diffusion-wave equation // Matimyás Matematika. – 1998. – **21**. – P. 109–118.
8. *Gorenflo R., Mainardi F., Moretti D., Paradisi P.* Time fractional diffusion: a discrete random walk approach // Nonlinear Dynam. – 2002. – **29**, No. 1-4. – P. 129–143.
9. *Kemppainen J.* Existence and uniqueness of the solution for a time-fractional diffusion equation with Robin boundary condition // Abstr. Appl. Anal. – 2011. – Article ID 321903. – 11 pp.
10. *Kilbas A. A., Srivastava H. M., Trujillo J. J.* Theory and applications of fractional differential equations. – Amsterdam: Elsevier, 2006. – 523 p.
11. *Magin R. L.* Fractional calculus in bioengineering. – Connecticut: Begell House Publ., Inc, 2006. – 684 p.
12. *Mainardi F.* Fractional relaxation-oscillation and fractional diffusion-wave phenomena // Chaos, Solitons Fract. – 1996. – **7**, No. 9. – P. 1461–1477.
13. *Mainardi F.* The fundamental solutions for the fractional diffusion-wave equation // Appl. Math. Lett. – 1996. – **9**, No. 6. – P. 23–28.
14. *Povstenko Y. Z.* Central symmetric solution to the Neumann problem for a time-fractional diffusion-wave equation in a sphere // Nonlinear Anal.: Real World Appl. – 2012. – **13**, No. 3. – P. 1229–1238.

15. *Povstenko Y. Z.* Fractional Cattaneo-type equations and generalized thermoelasticity // J. Therm. Stresses. – 2011. – **34**, No. 2. – P. 97–114.
16. *Povstenko Y. Z.* Fractional heat conduction equation and associated thermal stresses // J. Therm. Stresses. – 2005. – **28**, No. 1. – P. 83–102.
17. *Povstenko Y.* Non-axisymmetric solutions to time-fractional diffusion-wave equation in an infinite cylinder // Fract. Calc. Appl. Anal. – 2011. – **14**, No. 3. – P. 418–435.
18. *Povstenko Y. Z.* Nonaxisymmetric solutions of the time-fractional heat conduction equation in a half-space in cylindrical coordinates // J. Math. Sci. – 2012. – **183**, No. 2. – P. 252–260.
    Povstenko Y. Z. Non-axisymmetric solutions to time-fractional heat conduction equation in a half-space in cylindrical coordinates // Мат. методи та фіз.-мех. поля. – 2011. – **54,** № 1. – С. 212–219.
19. *Povstenko Y. Z.* Signaling problem for time-fractional diffusion-wave equation in a half-plane // Fract. Calc. Appl. Anal. – 2008. – **11**, No. 3. – P. 329–352.
20. *Povstenko Y. Z.* Signaling problem for time-fractional diffusion-wave equation in a half-space in the case of angular symmetry // Nonlinear Dynam. – 2010. – **59**, No. 4. – P. 593–605.
21. *Povstenko Y. Z.* Theory of thermoelasticity based on the space-time-fractional heat conduction equation // Phys. Scr. – 2009. – **T136**. – 014017 (6 pp).
22. *Povstenko Y. Z.* Thermoelasticity which uses fractional heat conduction equation // J. Math. Sci. – 2009. – **162**, No. 2. – P. 296–305.
    Povstenko Y. Z. Thermoelasticity which uses fractional heat conduction equation // Мат. методи та фіз.-мех. поля. – 2008. – **51,** № 2. – С. 239–246.
23. *Schneider W. R., Wyss W.* Fractional diffusion and wave equations // J. Math. Phys. – 1989. – **30**, No. 1. – P. 134–144.
24. *West B. J., Bologna M., Grigolini P.* Physics of fractal operators. – New York: Springer, 2003. – 354 p.
25. *Wyss W.* The fractional diffusion equation // J. Math. Phys. – 1986. – **27**, No. 11. – P. 2782–2785.

ФУНДАМЕНТАЛЬНІ РОЗВ'ЯЗКИ РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ З ДРОБОВОЮ ПОХІДНОЮ ЗА ЧАСОМ З ГРАНИЧНОЮ УМОВОЮ РОБЕНА ДЛЯ ПІВПРЯМОЇ

Розглянуто рівняння теплопровідності з похідною Капуто за часом дробовою порядку $0 < \alpha \le 2$ *у випадку півпрямої. Досліджено два типи граничної умови Робена: математичну умову, коли на границі задано лінійну комбінацію температури та її нормальної похідної, а також фізичну умову, коли на границі задано лінійну комбінацію температури та теплового потоку. Ці два типи граничної умови Робена співпадають тільки у випадку класичного рівняння теплопровідності.*

ФУНДАМЕНТАЛЬНЫЕ РЕШЕНИЯ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С ДРОБНОЙ ПРОИЗВОДНОЙ ПО ВРЕМЕНИ С ГРАНИЧНЫМ УСЛОВИЕМ РОБЕНА ДЛЯ ПОЛУПРЯМОЙ

Рассматривается уравнение теплопроводности с производной Капуто по времени дробного порядка $0 < \alpha \le 2$ *в случае полупрямой. Исследуются два типа граничного условия Робена: математическое условие, когда на границе задана линейная комбинация температуры и ее нормальной производной, а также физическое условие, когда на границе задана линейная комбинация температуры и теплового потока. Эти два типа граничных условий Робена совпадают только в случае классического уравнения теплопроводности.*

Inst. Math. and Comput. Sci.
Jan Długosz Univ. in Częstochowa, Poland
Received
31.12.11