UDC 539.3

A. Ya. Grigorenko[1], W. H. Müller[1,2], R. Wille[1,2], S. N. Yaremchenko[1,2]

NUMERICAL SOLUTION OF THE PROBLEM ON THE STRESS-STRAIN STATE IN HOLLOW CYLINDERS BY MEANS OF SPLINE-APPROXIMATIONS

Three-dimensional theory of elasticity is used for a study of the stress-strain state in an hollow cylinder with varying stiffness. The corresponding problem is solved by a method which is partly analytical and partly numerical in nature: Spline approximations and collocation are used to reduce the partial differential equations of elasticity to a boundary-value problem for a system of ordinary differential equations of higher order for the radial coordinate, which is then solved by using the method of stable discrete orthogonalization. Results for an inhomogeneous cylinder for various types of stiffness are presented.

The increasingly stringent requirements for estimation of strength characteristics, the tendency toward a detailed consideration of real properties of structural materials, and the discovery and study of three-dimensional effects occurring in thick-walled elements require the treatment of hollow cylindrical structures in terms of a three-dimensional model. Finding a solution for the stress-strain state in thick-walled structures in the framework of spatial linear elasticity theory goes hand-in-hand with significant difficulties related to the complexity of the initial systems and of the partial differential equations, as well as the necessity to satisfy the boundary conditions prescribed on the surfaces of the elastic body. These difficulties rise substantially during the calculation of structural elements such as cylinders made of anisotropic and inhomogeneous materials. The facts mentioned above are consistent with the relative sparseness of the number of publications addressing such questions (L. P. Kollar, J. M. Patterson, G. S. Springer [11], P. K. Banerjee, D. P. Henry [3], L. P. Kollar [10], Z. Shi, T. Zhang, H. Xiang [12], D. Gal, J. Dvorkin [5], F. Collin, D. Caillerie, R. Chamlon [4], I. Tsurkov, B. Drach [13]).

Along with universal approaches used for solving boundary value problems in mechanics and mathematical physics, such as the finite-difference technique, finite-elements, and other discrete methods, a new technique now finds wide application for this particular class of problems [2, 3, 14]. It allows reducing the initial problem to a system of ordinary differential equations, based on an approximation of the solution with respect to other variables by analytical methods. The exact reduction of multi-dimensional problems to one-dimensional ones and the solution of the latter by the stable numerical method of discrete orthogonalization gives reasons to believe that the obtained results are highly accurate. Due to the cylindrical geometry the method of finite-elements, if used for calculating the mechanical behavior, is time-consuming, ineffective, and requires large memory and processing speed of the computer.

Recently an approach based on spline-approximations was developed in several articles [6–9] in order to study the mechanical behavior of plate and shells. Its main advantages are [1]:

- stability against local perturbations, *i.e.*, the local behavior of splines in the neighborhood of a point does not influence their overall behavior, in contrast to, for example, polynomial approximation;
- better convergence than that of polynomial approximation;
- simple and convenient computer implementation.

The main goal of this article is the development of an efficient numerical-analytical approach to the solution of the problems for finding the stress-strain state states of hollow composite cylinders in a three-dimensional loading case. The proposed approach is a discrete-continuous one, and based on the combination of the spline-collocation method with the method of discrete-orthogonalization. It allows reducing three-dimensional problems to one-dimensional ones and solving the latter by the stable numerical method of discrete orthogonalization with a high degree of accuracy.

Basic Equations. We consider a hollow orthotropic cylinder of constant thickness (Fig. 1), inner radius $R - H$, outer radius $R + H$ (R is the radius of the mid-surface; $2H$ is the thickness of the cylinder), and length L, described in a cylindrical coordinate system r, θ, z. The stress-strain state of such a cylinder is described by the following equations of elasticity:

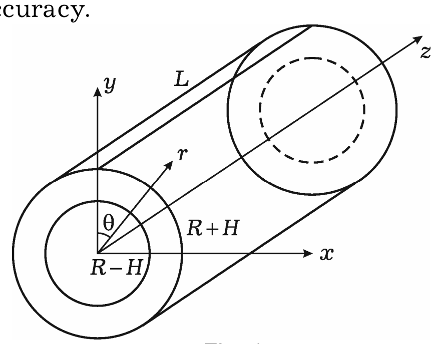

Fig. 1

- linear kinematic relations

$$e_r = \frac{\partial u_r}{\partial r}, \qquad e_z = \frac{\partial u_z}{\partial z}, \qquad 2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}; \tag{1}$$

- Hooke's law for the more general orthotropic case

$$\sigma_r = \lambda_{11} e_r + \lambda_{12} e_\theta + \lambda_{13} e_z,$$
$$\sigma_\theta = \lambda_{12} e_r + \lambda_{22} e_\theta + \lambda_{23} e_z,$$
$$\sigma_z = \lambda_{13} e_r + \lambda_{23} e_\theta + \lambda_{33} e_z, \tag{2}$$

where the elements $\lambda_{ij} = \lambda_{ij}(r,z)$ of the stiffness matrix are continuous and differentiable functions of the coordinates r and z;

- equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \qquad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0, \tag{3}$$

$u_r(r,z)$, $u_z(r,z)$ are the projections of the total displacement of the cylinder onto the tangents to the coordinate lines r and z, respectively; e_r, e_θ, e_z are the relative linear strains along the coordinate lines; e_{rz} is the shear strain; σ_r, σ_θ, σ_z are the normal stresses; σ_{rz} is the tangential stress.

The elements λ_{ij} of the stiffness matrix follow from the elements c_{ij} of the compliance matrix as

$$\lambda_{11} = (\tilde{c}_{22}\tilde{c}_{33} - \tilde{c}_{23}^2)\frac{1}{\Delta}, \qquad \lambda_{12} = (\tilde{c}_{13}\tilde{c}_{23} - \tilde{c}_{12}\tilde{c}_{33})\frac{1}{\Delta},$$
$$\lambda_{13} = (\tilde{c}_{12}\tilde{c}_{23} - \tilde{c}_{13}\tilde{c}_{22})\frac{1}{\Delta}, \qquad \lambda_{22} = (\tilde{c}_{11}\tilde{c}_{33} - \tilde{c}_{13}^2)\frac{1}{\Delta},$$
$$\lambda_{23} = (\tilde{c}_{12}\tilde{c}_{13} - \tilde{c}_{11}\tilde{c}_{23})\frac{1}{\Delta}, \qquad \lambda_{33} = (\tilde{c}_{11}\tilde{c}_{22} - \tilde{c}_{12}^2)\frac{1}{\Delta}, \qquad \lambda_{55} = \frac{1}{c_{55}},$$
$$\Delta = c_{11}(c_{22}c_{33} - c_{23}^2) - c_{12}(c_{12}c_{33} - c_{13}c_{23}) + c_{13}(c_{12}c_{23} - c_{13}c_{22}). \tag{4}$$

In turn, the elements of the compliance matrix can be expressed in terms of the engineering constants:

$$c_{11} = \frac{1}{E_r}, \qquad c_{12} = -\frac{\nu_{r\theta}}{E_\theta}, \qquad c_{13} = -\frac{\nu_{rz}}{E_z},$$
$$c_{22} = \frac{1}{E_\theta}, \qquad c_{23} = -\frac{\nu_{\theta z}}{E_z}, \qquad c_{33} = \frac{1}{E_z}, \qquad c_{55} = -\frac{1}{G_{rz}}, \tag{5}$$

where E_r, E_θ, E_z are the elastic moduli in the r-, θ-, and z-directions, respectively, G_{rz} is the shear modulus; and $\nu_{r\theta}$, ν_{rz}, $\nu_{\theta z}$ are Poisson's ratios.

The boundary conditions on the internal $R - H$ and external $R + H$ surfaces of the cylinder are given by

$$\sigma_r(R - H, z) = 0, \qquad \sigma_r(R + H, z) = q, \qquad \sigma_{rz}(R \pm H, z) = 0\,. \tag{6}$$

We prescribe the following boundary conditions at the ends $z = 0$ and $z = L$:

$$(\boldsymbol{i}) \qquad \sigma_r = 0, \qquad u_r = 0 \qquad \text{or} \qquad \frac{\partial u_z}{\partial z} = 0, \qquad u_r = 0\,; \tag{7}$$

$$(\boldsymbol{ii}) \qquad u_z = 0, \qquad \sigma_{rz} = 0 \qquad \text{or} \qquad u_z = 0, \qquad \frac{\partial u_r}{\partial z} = 0\,; \tag{8}$$

$$(\boldsymbol{iii}) \qquad u_r = 0, \qquad u_z = 0\,. \tag{9}$$

The following system of equations for the displacements results

$$\begin{aligned}
\frac{\partial^2 u_r}{\partial r^2} &= \left(-\frac{1}{\lambda_{11}}\frac{\partial \lambda_{12}}{\partial r}\frac{1}{r} + \frac{\lambda_{22}}{\lambda_{11}}\frac{1}{r^2}\right)u_r - \frac{1}{\lambda_{11}}\frac{\partial \lambda_{55}}{\partial z}\frac{\partial u_r}{\partial z} - \frac{\lambda_{55}}{\lambda_{11}}\frac{\partial^2 u_r}{\partial z^2} - \\
&\quad - \left(\frac{1}{\lambda_{11}}\frac{\partial \lambda_{11}}{\partial r} + \frac{1}{r}\right)\frac{\partial u_r}{\partial r} - \left(\frac{1}{\lambda_{11}}\frac{\partial \lambda_{13}}{\partial r} - \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11}}\frac{1}{r}\right)\frac{\partial u_z}{\partial z} - \\
&\quad - \frac{1}{\lambda_{11}}\frac{\partial \lambda_{55}}{\partial z}\frac{\partial u_z}{\partial r} - \frac{\lambda_{13} + \lambda_{55}}{\lambda_{11}}\frac{\partial^2 u_z}{\partial z \partial r}, \\
\frac{\partial^2 u_z}{\partial r^2} &= -\frac{1}{\lambda_{55}}\frac{\partial \lambda_{23}}{\partial z}\frac{u_r}{r} - \left(\frac{1}{\lambda_{55}}\frac{\partial \lambda_{55}}{\partial r} + \frac{\lambda_{23}}{\lambda_{55}}\frac{1}{r} + \frac{1}{r}\right)\frac{\partial u_r}{\partial z} - \left(1 + \frac{\lambda_{13}}{\lambda_{55}}\right)\frac{\partial^2 u_r}{\partial r \partial z} - \\
&\quad - \frac{1}{\lambda_{55}}\frac{\partial \lambda_{13}}{\partial z}\frac{\partial u_r}{\partial r} - \frac{1}{\lambda_{55}}\frac{\partial \lambda_{33}}{\partial z}\frac{\partial u_z}{\partial z} - \\
&\quad - \frac{\lambda_{33}}{\lambda_{55}}\frac{\partial^2 u_z}{\partial z^2} - \left(\frac{1}{r} + \frac{1}{\lambda_{55}}\frac{\partial \lambda_{55}}{\partial r}\right)\frac{\partial u_z}{\partial r}\,.
\end{aligned} \tag{10}$$

We now reduce these equations to the form

$$\begin{aligned}
\frac{\partial^2 u_r}{\partial r^2} &= a_{11}u_r + a_{12}\frac{\partial u_r}{\partial z} + a_{13}\frac{\partial^2 u_r}{\partial z^2} + a_{14}\frac{\partial u_r}{\partial r} + \\
&\quad + a_{15}\frac{\partial u_z}{\partial z} + a_{16}\frac{\partial u_z}{\partial r} + a_{17}\frac{\partial^2 u_z}{\partial r \partial z}, \\
\frac{\partial^2 u_z}{\partial r^2} &= a_{21}u_r + a_{22}\frac{\partial u_r}{\partial z} + a_{23}\frac{\partial u_r}{\partial r} + a_{24}\frac{\partial^2 u_r}{\partial r \partial z} + \\
&\quad + a_{25}\frac{\partial u_z}{\partial z} + a_{26}\frac{\partial^2 u_z}{\partial z^2} + a_{27}\frac{\partial u_z}{\partial r},
\end{aligned} \tag{11}$$

where the coefficients $a_{k\ell} = a_{k\ell}(r, z)$ are defined by

$$a_{11} = -\frac{1}{\lambda_{11}}\frac{\partial \lambda_{12}}{\partial r}\frac{1}{r} + \frac{\lambda_{22}}{\lambda_{11}}\frac{1}{r^2}, \qquad a_{12} = -\frac{1}{\lambda_{11}}\frac{\partial \lambda_{55}}{\partial z},$$

$$a_{13} = -\frac{\lambda_{55}}{\lambda_{11}}, \qquad a_{14} = -\left(\frac{1}{\lambda_{11}}\frac{\partial \lambda_{11}}{\partial r} + \frac{1}{r}\right),$$

$$a_{15} = -\left(\frac{1}{\lambda_{11}}\frac{\partial \lambda_{13}}{\partial r} - \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11}}\frac{1}{r}\right), \qquad a_{16} = -\frac{1}{\lambda_{11}},$$

$$\frac{\partial\lambda_{55}}{\partial z} a_{17} = -\frac{\lambda_{13} + \lambda_{55}}{\lambda_{11}}, \qquad a_{21} = -\frac{1}{\lambda_{55}}\frac{\partial\lambda_{23}}{\partial z}\frac{1}{r},$$

$$a_{22} = -\left(\frac{1}{\lambda_{55}}\frac{\partial\lambda_{55}}{\partial r} + \frac{\lambda_{23}}{\lambda_{55}}\frac{1}{r} + \frac{1}{r}\right), \qquad a_{23} = -\frac{1}{\lambda_{55}}\frac{\partial\lambda_{13}}{\partial z},$$

$$a_{24} = -\left(1 + \frac{\lambda_{13}}{\lambda_{55}}\right), \qquad a_{25} = -\frac{1}{\lambda_{55}}\frac{\partial\lambda_{33}}{\partial z},$$

$$a_{26} = -\frac{\lambda_{33}}{\lambda_{55}}, \qquad a_{27} = -\left(\frac{1}{r} + \frac{1}{\lambda_{55}}\frac{\partial\lambda_{55}}{\partial r}\right). \tag{12}$$

In this case the boundary conditions (6) on the inner and outer surfaces become

$$\lambda_{11}\frac{\partial u_r}{\partial r} + \lambda_{12}\frac{u_r}{r} + \lambda_{13}\frac{\partial u_z}{\partial z} = 0, \qquad \lambda_{55}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0. \tag{13}$$

Solving technique. The problem defined by Eqn. (10) in combination with appropriate boundary conditions can be solved by spline-collocation and discrete-orthogonalization methods. In preparation for the spline-collocation method we write the unknown functions $u_r(r,z)$, $u_z(r,z)$ as follows:

$$u_r = \sum_{i=0}^{N} u_{ri}(r)\varphi_i^{(1)}(z), \qquad u_z = \sum_{i=0}^{N} u_{zi}(r)\varphi_i^{(2)}(z), \tag{14}$$

where $u_{ri}(r)$, $u_{zi}(r)$ are sought functions of the variable r, $\varphi_i^{(j)}(z)$, $j = 1,2$, $i = 0,1,\ldots,N$, are linear combinations of B-splines on the uniform mesh Δ: $0 = z_0 < z_1 < \ldots < z_N = L$ which must satisfy the boundary conditions at $z = 0$ and $y = L$. The system (10) includes derivatives of the unknown functions along the coordinate z no higher than second order. In this case, we may restrict ourselves with approximations of third power, *i.e.*

$$B_3^i(z) = \frac{1}{6}\begin{cases} 0, & -\infty < z < z_{i-2}, \\ y^3, & z_{i-2} \le z < z_{i-1}, \\ -3y^2 + 3y^2 + 3y + 1, & z_{i-1} \le z < z_i, \\ 3y^3 - 6y^2 + 4, & z_i \le z < z_{i+1}, \\ (1-y)^3, & z_{i+1} \le z < z_{i+2}, \\ 0, & z_{i+2} \le z < \infty, \end{cases} \tag{15}$$

where $y = \frac{z - z_k}{h_z}$ on the interval $[z_k, z_{k+1}]$, $k = i-2,\ldots,i+1$, $i = -1,\ldots,N+1$; $h_z = z_{k+1} - z_k = \text{const}$. In this case, the functions $\varphi_i^{(j)}(z)$ are as follows:

1°) If the relevant resolving function (u_r or u_z) at $z = 0$ and $z = L$ is equal to zero, then

$$\varphi_0^{(j)}(z) = -4B_3^{-1}(z) + B_3^0(z), \qquad \varphi_1^{(j)}(z) = B_3^{-1}(z) - \frac{1}{2}B_3^0(z) + B_3^1(z),$$

$$\varphi_i^{(j)}(z) = B_3^i(z), \qquad i = 2,3,\ldots,N-2,$$

$$\varphi_{N-1}^{(j)}(z) = B_3^{N-1}(z) - \frac{1}{2}B_3^N(z) + B_3^{N+1}(z),$$

$$\varphi_N^{(j)}(z) = -4B_3^{N+1}(z) + B_3^N(z); \tag{16}$$

2°) If the derivative with respect to the resolving function at $z = 0$ and $z = L$ is equal to zero, then

$$\varphi_0^{(j)}(z) = B_3^0(z), \qquad \varphi_1^{(j)}(z) = B_3^{-1}(z) - \frac{1}{2}B_3^0(z) + B_3^1(z),$$

$$\varphi_{ji}(z) = B_3^i(z), \qquad i = 2,3,\ldots,N-2,$$

$$\varphi_{N-1}^{(j)}(z) = B_3^{N-1}(z) - \frac{1}{2}B_3^N(z) + B_3^{N+1}(z), \qquad \varphi_N^{(j)}(z) = B_3^N(z); \qquad (17)$$

3°) If the relevant resolving function at $z = 0$ is equal to zero and at $z = L$ the derivative with respect to z of the resolving function is also equal to zero, then

$$\varphi_0^{(j)}(z) = -4B_3^{-1}(z) + B_3^0(z), \qquad \varphi_1^{(j)}(z) = B_3^{-1}(z) - \frac{1}{2}B_3^0(z) + B_3^1(z),$$

$$\varphi_i^{(j)}(z) = B_3^i(z), \qquad i = 2,3,\ldots,N-2,$$

$$\varphi_{N-1}^{(j)}(z) = B_3^{N-1}(z) - \frac{1}{2}B_3^N(z) + B_3^{N+1}(z), \qquad \varphi_N^{(j)}(z) = B_3^N(z). \qquad (18)$$

By substituting Eqns. (14) into (10), we now require them to be satisfied at the specified collocation points $\xi_k \in [0,L]$, $k = \overline{0,N}$. We consider the case when the number of mesh nodes is even, *i.e.*, $N = 2n+1$, $n \ge 3$. The selection of the collocation points $\xi_{2i} \in [z_{2i}, z_{2i+1}]$, $\xi_{2i+1} \in [z_{2i}, z_{2i+1}]$, $i = 0,1,2,\ldots,n$, in the form $\xi_{2i} = z_{2i} + s_1 h_z$, $\xi_{2i+1} = z_{2i} + s_2 h_z$, where $s_1 = 1/2 - \sqrt{3}/6$, $s_2 = 1/2 + \sqrt{3}/6$, are the roots of the second-order Legendre polynomial, is optimal and essentially increases the degree of accuracy of the approximation. In this case, the number of collocation points is $\bar{N} = N+1$. As a result, we obtain a system of $4(N+1)$ linear differential equations with respect to the functions u_{ri}, $\tilde{u}_{ri}$, u_{zi}, $\tilde{u}_{zi}$, $i = 0,\ldots,N$, where $u'_{ri} = \tilde{u}_{ri}$, $u'_{zi} = \tilde{u}_{zi}$. By employing the following notation

$$\Phi_j = [\varphi_i^{(j)}(\xi_k)], \qquad k,i = 0,\ldots,N, \qquad j = 1,2,$$

$$\bar{u}_r = \{u_{r0}, u_{r1}, \ldots, u_{rN}\}^\top, \qquad \bar{\tilde{u}}_r = \{\tilde{u}_{r0}, \tilde{u}_{r1}, \ldots, \tilde{u}_{rN}\}^\top,$$

$$\bar{u}_z = \{u_{z0}, u_{z1}, \ldots, u_{zN}\}^\top, \qquad \bar{\tilde{u}}_z = \{\tilde{u}_{z0}, \tilde{u}_{z1}, \ldots, \tilde{u}_{zN}\}^\top,$$

$$\bar{a}_{k\ell}^\top = \{a_{k\ell}(r,\xi_0),\ a_{k\ell}(r,\xi_1), \ldots, a_{k\ell}(r,\xi_N)\}, \qquad (19)$$

and by designating the matrix $[c_i a_{ij}]$ in the form $\bar{c} * A$ for the matrix $A = [a_{ij}]$, $i,j = 0,\ldots,N$ and the vector $\bar{c} = \{c_0, c_1, \ldots, c_N\}^\top$, the system of ordinary differential equations with respect to u_{ri}, $\tilde{u}_{ri}$, u_{zi}, $\tilde{u}_{zi}$ takes on the form

$$\frac{d\bar{u}_r}{dr} = \bar{\tilde{u}}_r,$$

$$\frac{d\bar{u}_z}{dr} = \bar{\tilde{u}}_z,$$

$$\frac{d\bar{\tilde{u}}_r}{dr} = \Phi_1^{-1}(\bar{a}_{11} * \Phi_1 + \bar{a}_{12} * \Phi_1' + \bar{a}_{13} * \Phi_1'')\bar{u}_r + \Phi_1^{-1}(\bar{a}_{14} * \Phi_1)\bar{\tilde{u}}_r +$$

$$+ \Phi_1^{-1}(\bar{a}_{15} * \Phi_2')\bar{u}_z + \Phi_1^{-1}(\bar{a}_{16} * \Phi_2 + \bar{a}_{17} * \Phi_2')\bar{\tilde{u}}_z,$$

$$\frac{d\bar{\tilde{u}}_z}{dy} = \Phi_2^{-1}(\bar{a}_{21}\Phi_1 + \bar{a}_{22}\Phi_1')\bar{u}_r + \Phi_2^{-1}(\bar{a}_{23} * \Phi_1')\bar{\tilde{u}}_r +$$

$$+ \Phi_2^{-1}(\bar{a}_{24} * \Phi_2 + \bar{a}_{25} * \Phi_2' + \bar{a}_{26} * \Phi_2'')\bar{u}_z + \Phi_2^{-1}(\bar{a}_{27} * \Phi_2)\bar{\tilde{u}}_z, \quad (20)$$

which can be represented as

$$\frac{d\bar{Y}}{dr} = A(r)\bar{Y}, \qquad R - H \le r \le R + H, \quad (21)$$

where

$$\bar{Y} = \{u_{r0},\ldots,u_{rN},\ \tilde{u}_{r0},\ldots,\tilde{u}_{rN},\ u_{z0},\ldots,u_{zN},\ \tilde{u}_{zo},\ldots,\tilde{u}_{zN}\}^{\top}$$

is a vector-function, depending on r, and $A(r)$ is a square matrix of $4(N+1) \times 4(N+1)$ th-order.

Boundary conditions for this system of ordinary differential equations are defined by

$$\bar{\lambda}_{11}\Phi_1\bar{\tilde{u}}_r + \bar{\lambda}_{12}\Phi_1\frac{1}{r}\bar{u}_r + \bar{\lambda}_{13}\Phi_2'\bar{u}_z = \bar{q}, \qquad \bar{\lambda}_{55}\Phi_1'\bar{u}_r + \bar{\lambda}_{55}\Phi_2\bar{\tilde{u}}_z = 0, \quad (22)$$

where

$$\bar{\lambda}_{1\ell}^{\top} = \{\lambda_{1\ell}(r,\xi_0),\lambda_{1\ell}(r,\xi_1),\ldots,\lambda_{1\ell}(r,\xi_N)\}, \qquad \ell = 1,2,3,$$

$$\bar{\lambda}_{55}^{\top} = \{\lambda_{55}(r,\xi_0),\lambda_{55}(r,\xi_1),\ldots,\lambda_{55}(r,\xi_N)\},$$

or by

$$B_1\bar{Y}(R-H) = \bar{b}_1, \qquad B_2\bar{Y}(R+H) = \bar{0}, \quad (23)$$

where B_1 and B_2 are rectangular matrices of the $2(N+1) \times 4(N+1)$ th-order, b_1 is corresponding vector.

The boundary-value problem (21), (23) can be solved using a discrete orthogonalization method.

Numerical results. The modulus of elasticity E is supposed to vary along the radial coordinate r according to a power law

$$E(r) = \frac{E_0}{1+\alpha}\left(1 + \alpha\left(\frac{r}{R-H}\right)^{\beta}\right). \quad (24)$$

The following parameters were used in context with the cylinder: $L = 10$, $R = 10$, $H = 1$. Poisson's ratio $\nu = 0.34$. The ends of cylinder are clamped.

The dependencies of the radial displacement $u_r = u_r E_0/q$ and the circumferential stress $\sigma_\theta = \sigma_\theta/q$ on the parameters used for the variation of Young's modulus (see Eq. (24)) are shown in Figs 2–5 ($\alpha = 1$ for varying values of β, and $\beta = 1$ for varying values of α). The displacements and stresses in the middle section of the cylinder, *i.e.*, at $z = L/2$ are shown for the inner surface at $r = R - H$ (solid lines), for $r = R - H/2$ (dashed line), for $r = R$ (dotted line), for $r = R + H/2$ (dashed-dotted line), and on the outer surface at $R = R + H$ (dashed-double dotted line).

The radial displacement $\hat{u}_r$ decreases when the parameter β increases from -5 to 5 (Fig. 2). The difference between the displacements on the inner and on the outer surfaces decreases with increasing β.

Fig. 3 shows that circumferential stress $\hat{\sigma}_\theta$ on the inner surface decreases with increasing β. On the outer surface it behaves in the opposite way. In contrast to that the stress in the mid-surface (when $r = R$) changes only slightly. Also, when β is negative, the circumferential stress on the inner surface is greater than on the outer surface, and vice versa for positive values of β.

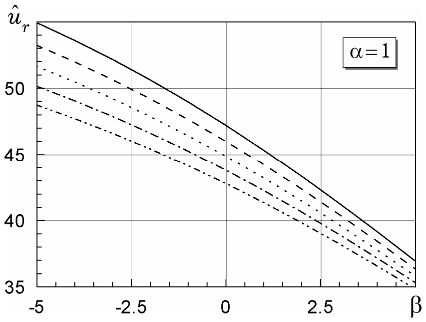

Fig. 2. The displacement $\hat{u}_r$ and its dependence on the parameter β

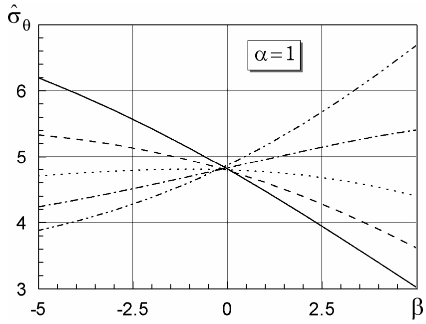

Fig. 3. The stress $\hat{\sigma}_\theta$ and its dependence on the parameter β

From Fig. 4 it becomes evident that the displacement u_r decreases when the parameter α increases from 0 to 10. The figure also shows that the greatest changes in displacement occur within the interval $0 \le \alpha \le 5$, whereas for $5 \le \alpha \le 10$ the displacement varies only slightly. The same effect is observed in case of the stress (Fig. 5). Moreover the circumferential stress $\hat{\sigma}_\theta$ increases on the outer surface and decreases on the inner surface for increasing α, just as in the case of β. Also the maximum circumferential stress $\hat{\sigma}_\theta$ shifts from the inner surface to the outer with increasing α.

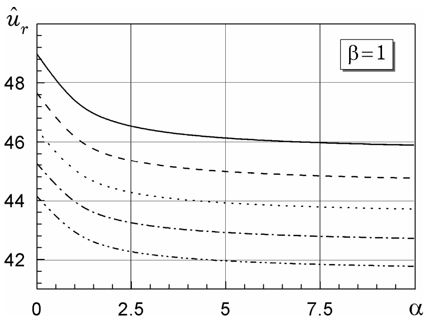

Fig. 4. The displacement $\hat{u}_r$ and its dependence on the parameter α.

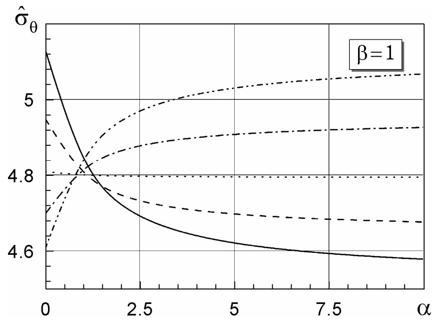

Fig. 5. The stress $\hat{\sigma}_\theta$ and its dependence on the parameter α.

Fig. 6 shows how the radial stress $\hat{\sigma}_r = \sigma_r/q$ varies from the inner to the outer cylinder surface in the middle section of the cylinder ($z = L/2$) depending on the value of β parameter. The following designations were used: solid line for $\beta = -10$, dashed line for $\beta = -5$, dotted line for $\beta = 0$, dashed-dotted line for $\beta = 5$, dashed-double dotted line for $\beta = 10$ $(\alpha = 1)$. Predictably (as evident from the boundary conditions), the radial stresses on the inner and on the outer surfaces are -1 and 0, respectively. The curves change from concave to convex for increasing β. The dependence of the stresses on the radius comes close to a straight line for $\beta = 5$.

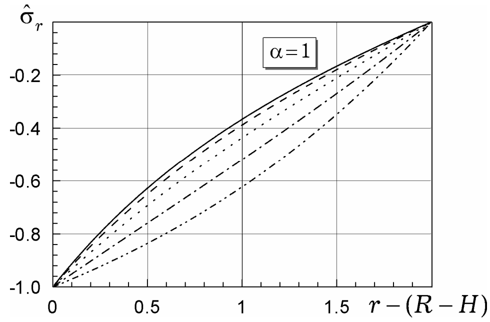

Fig. 6. Radial stress $\hat{\sigma}_\theta$ distribution as a function of radius r for various values of β

Thus by varying the physical parameters of the construction material it is possible to influence the stress-strain distribution within the cylinder and to choose optimum parameters for its strength.

1. *Ahlberg J. H., Nilson E. N., Walsh J. L.* The theory of splines and their applications. – New York: Acad. Press, 1967. – 284 p.
2. *Babuška I., Banerjee U., Osborn J. E.* Generalized finite element methods – main ideas, results and perspective // Int. J. Comput. Methods. – 2004. – **1**. – P. 67–103.
3. *Banerjee P. K., Henry D. P.* Elastic analysis of three-dimensional solids with fiber inclusions by BEM // Int. J. Solids and Struct. – 1992. – **29**, No. 20. – P. 2423–2440.
4. *Collin F., Caillerie D., Chambon R.* Analytical solutions for the thick-walled cylinder problem modeled with an isotropic elastic second gradient constitutive equation // Int. J. Solids and Struct. – 2009. – **46,** No. 22-23. – P. 3927–3937.
5. *Gal D., Dvorkin J.* Stresses in anisotropic cylinders // Mech. Res. Commun. – 2009. – **22**. – P. 109–113.
6. *Grigorenko Ya. M., Yaremchenko S. N.* Refined analysis of the stress state of orthotropic elliptic cylindrical shells with variable geometrical parameters // Int. Appl. Mech. – 2008. – **44**, No. 9. – P. 998–1005.
7. *Grigorenko A., Yaremchenko S.* Investigation of static and dynamic behavior of anisotropic inhomogeneous shallow shells by spline approximation method // J. Civil Eng. and Manag. – 2009. – **15**, No. 1. – P. 87–93.
8. *Grigorenko Ya. M., Grigorenko A. Ya. Zakhairichenko L. I.* Influence of geometrical parameters on the stress state of longitudinally corrugated elliptic cylindrical shells // Int. Appl. Mech. – 2009. – **45**, No. 2. – P. 187–192.
9. *Grigorenko A. Ya., Yaremchenko N. P., Yaremchenko S. N.* Spline-based investigation of stress-strain state of anisotropic rectangular shallow shells of variable thickness in refined formulation // J. Mech. of Mater. and Struct. (Submitted in 2009).
10. *Kollar L. P.* Three-dimensional analysis of composite cylinders under axially varying hydrothermal and mechanical loads // Comput. & Struct. – 1994. – **50**, No. 4. – P. 525–540.
11. *Kollar L. P., Patterson J. M., Springer G. S.* Composite cylinders subjected to hydrothermal and mechanical loads // Int. J. Solids and Struct. – 1992. – **29**, No. 12. – P. 1519–1534.
12. *Shi Z., Zhang T., Xiang H.* Exact solution of heterogeneous elastic hollow cylinders // Composite Struct. – 2007. – **79**, No. 1. – P. 140–147.
13. *Tsurkov I., Drach B.* Elastic deformation of composite cylinders with cylindrically orthotropic layers // Int. J. Solids and Struct. – 2010. – **47**, No. 1. – P. 25–33.
14. *Zienkiewicz O. C., Taylor R. L.* The finite element method for solid and structural mechanics. – Elsevier, 2005. – 685 p.

ЧИСЕЛЬНЕ РОЗВ'ЯЗАННЯ ЗАДАЧІ ПРО НАПРУЖЕНИЙ СТАН ПОРОЖНИСТИХ ЦИЛІНДРІВ ЗА ДОПОМОГОЮ СПЛАЙН-АПРОКСИМАЦІЇ

Для дослідження напруженого стану порожнистих циліндрів зі змінною жорсткістю використано тривимірну теорію пружності. Задачу, що розглядається, розв'язано чисельно аналітичним методом: на першому етапі використовується сплайн-апроксимація і метод колокації для зведення задачі в частинних похідних до одновимірної крайової задачі високого порядку в радіальному напрямку. Отримана задача розв'язується стійким чисельним методом дискретної ортогоналізації. Наведено результати розв'язання задач для неоднорідного циліндра для різних варіантів зміни жорсткості.

ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧИ О НАПРЯЖЕННОМ СОСТОЯНИИ ПОЛЫХ ЦИЛИНДРОВ С ПОМОЩЬЮ СПЛАЙН-АППРОКСИМАЦИИ

Для исследования напряженного состояния полых цилиндров с переменной жесткостью использована трехмерная теория упругости. Рассмотренная задача решена численно аналитическим методом: на первом этапе используется сплайн-аппроксимация и метод коллокации для сведения задачи в частных производных к одномерной краевой задаче высокого порядка в радиальном направлении. Полученная задача решается устойчивым численным методом дискретной ортогонализации. Представлены результаты решения задач для неоднородного цилиндра для различных вариантов изменения жесткости.

[1] S. P. Timoshenko Inst. of Mechanics of NAS of Ukraine, Kiev,
[2] Technical Univ. of Berlin, Berlin (Germany)

Received
05.03.10