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DEFORMATION OF ORTHOTROPIC COMPOSITES WITH UNIDIRECTIONAL ELLIPSOIDAL INCLUSIONS UNDER MATRIX MICRODAMAGES

In the present paper a model of deformation of stochastic composites under microdamaging is developed for the case of orthotropic composite, when the microdamages are accumulated in the matrix. The composite is treated as an isotropic matrix strengthened by three-axial ellipsoidal inclusions with orthotropic symmetry of elastic properties. It is assumed that the loading process leads to accumulation of damages in the matrix. Fractured microvolumes are modelled by a system of randomly distributed quasi-spherical pores. The porosity balance equation and relations for determining the effective elastic modules for the case of the composite with orthotropic components are taken as the basic relations. The fracture criterion is assumed to be given as the limit value of the intensity of average shear stresses occurring in the undamaged part of the material. Basing on the analytical and numerical approach the algorithm for determination of nonlinear deformative properties of such a material is constructed. The nonlinearity of composite deformations is caused by accumulation of the micro damages in the matrix. Using the numerical solution the nonlinear stress-strain diagrams for orthotropic composite for the case of biaxial extension are obtained.

1. Introduction. The process of behaviour of composite materials for given significant loads represents large theoretical interest and is important for various applications. Some aspects of mechanic of composite material are considered in [1, 2, 5].

The determination of the strain-stress state diagram of a material by an experimental way is rather difficulty, therefore there is necessarily developing the various theoretical techniques of a research of the problem. The concept that the micro destruction occurs in the weakest micro volumes of a material under high loads, which reduces the bearing section of the material and leads to a redistribution of micro stresses, and hence to nonlinear relationships between macro stresses and macro deformations is based on the theory of material damage [3, 4, 6–8, 11]. In the statistical approaches it is used the assumption that the strength of the material is statistically homogeneous and is a random function of coordinates, the single-point distribution of which is described by a Weibull distribution [8, 11–14, 22, 23].

A statistical model of coupled processes of deformation and short-term micro damages was proposed for homogeneous [16, 17] and composite materials with isotropic [19, 20] and transversally-isotropic [18] components. It is based on modeling dispersed micro damages by a system of randomly distributed quasi spherical micro pores, which are empty or are filled with destroyed material. The process formation and accumulation of micro damages under loading is based on a failure criterion for micro volume in a Huber – Mises or in Schleicher – Nadai form and on the porosity balance of destroyed micro volumes derived from the general properties of distribution function for the ultimate micro strength. In the present work, it is developed a more general model within the same parent ideas, which allows to study the deformative properties of stochastic composites with isotropic matrix and orthotropic three axes ellipsoidal inclusions under matrix micro damages.

On the basis of the numerical and analytical approach it is constructed an algorithm for determination of the nonlinear deformative properties of such a material. The nonlinearity of composite deformations caused by accumulation of micro damages in matrix. Investigation of the nonlinear deformative properties of the composites may be divided in two stages: first, linear effective elastic properties of the porous composite is determined on the basis of the stochastic equations of elasticity theory taking into account an accidental character of the distribution of inclusions and micro pores in matrix by using the method of conditional moments [15], and then using the iterative secant method it is investigated the nonlinear deformative properties of such a material. The nonlinear diagrams of the deformation of orthotropic composite are constructed for the case of biaxial extension.

2. Mechanical model. Let us consider the representative volume V of a composite material subjected to uniform macro deformations $\langle \varepsilon_{k\ell} \rangle$. The composite is the matrix strengthened by stochastically distributed unidirectional three-axes ellipsoids as shown in Fig. 1. We suppose that the elastic properties of inclusions material have orthotropic symmetry. It is assumed, that the matrix is isotropic and weakened by randomly distributed pores of quasi spherical shape. The effective deformative properties and the stress-strain state of such a composite are determined on the basis of the stochastic equations of the elasticity theory by the method of conditional moment functions [15].

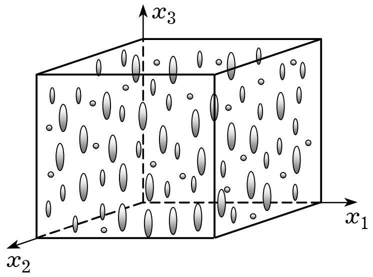

Fig. 1

Under homogeneous loading, the stresses and strains appearing in the representative volume will form statistically homogeneous random fields satisfying the ergodicity condition. In this case we can replace the operation of the averaging over a representative volume by the operation of the averaging over an ensemble of realizations. Then the macroscopic stresses $\langle \sigma_{ij} \rangle$ and strains $\langle \varepsilon_{k\ell} \rangle$ of such a material will be related as follows:

$$\langle \sigma_{ij} \rangle = \lambda^*_{ijk\ell} \langle \varepsilon_{k\ell} \rangle . \tag{1}$$

Here $\lambda^*_{ijk\ell}$ is the tensor of effective elastic constants, σ_{ij}, $\varepsilon_{k\ell}$ are the stresses and strains tensors, and the angular brackets denote the operation of the averaging over an ensemble of realizations. Assume that the matrix of such a composite has an initial porosity p_0. Then the tensor of effective elastic modules of composite will be function depending on the elastic modules of components $\lambda^{[1]}_{ijk\ell}$, $\lambda^{[2]}_{ijk\ell}$ (indexes [1] and [2] denote inclusions and matrix, respectively), on the volume concentration of inclusions c_1 in a matrix and the shape of inclusions characterized by parameters $\bar{t}_2$ and $\bar{t}_3$ [10] as:

$$\lambda^*_{ijk\ell} = \lambda^*_{ijk\ell}(\lambda^{[1]}_{mnpq}, \lambda^{[2]}_{mnpq}, c_1, \bar{t}_2, \bar{t}_3), \tag{2}$$

where

$$\bar{t}_2 = \frac{t_2}{t_1}, \qquad \bar{t}_3 = \frac{t_3}{t_1} . \tag{3}$$

Here t_1, t_2, t_3 are semi axes of ellipsoids in the directions of axes x_1, x_2, x_3 respectively.

The tensor $\lambda^{[2]}_{ijk\ell}$ are determined in terms of the tensor $\lambda^2_{ijk\ell}$ for the matrix skeleton and its porosity p_0, which characterizes damage, i.e.

$$\lambda^{[2]}_{ijk\ell} = \lambda^{[2]}_{ijk\ell}(\lambda^2_{mnpq}, p_0) . \tag{4}$$

The relations derived in [21] are used for calculation of the effective elastic constants of porous material.

Knowing the effective elastic modules and macro deformations of composite, we can determine the matrix deformations by the relations:

$$\langle \varepsilon_{k\ell} |_2 \rangle = \left(I_{k\ell pq} + c_2 \left(\langle \lambda_{k\ell mn} \rangle - \lambda^*_{k\ell mn} \right) (\lambda^{[3]}_{mnpq})^{-1} \right) \langle \varepsilon_{pq} \rangle . \tag{5}$$

Here

$$\langle \lambda_{k\ell mn} \rangle = c_1 \lambda^{[1]}_{k\ell mn} + c_2 \lambda^{[2]}_{k\ell mn}, \qquad \lambda^{[3]}_{mnpq} = \lambda^{[1]}_{mnpq} - \lambda^{[2]}_{mnpq}, \tag{6}$$

and c_1, c_2 are the volume concentrations of inclusion and binder in a composite and they are connected by following expression:

$$c_2 = 1 - c_1 . \tag{7}$$

The average matrix stresses $\langle \sigma_{ij} |_2 \rangle$ are related to the average matrix strains $\langle \varepsilon_{k\ell} |_2 \rangle$ as follows:

$$\langle \sigma_{ij} |_2 \rangle = \lambda^{[2]}_{ijk\ell} \langle \varepsilon_{k\ell} |_2 \rangle . \tag{8}$$

Whereas the average matrix skeleton stresses $\langle \sigma^2_{ij} \rangle$ are connected with the average matrix stresses $\langle \sigma_{ij} |_2 \rangle$ by equations

$$\langle \sigma^2_{ij} \rangle = \frac{1}{1 - p_0} \langle \sigma_{ij} |_2 \rangle \tag{9}$$

and regarding to Eqns. (1), (5)–(8), the average matrix skeleton stresses $\langle \sigma^2_{ij} \rangle$ can be determined as a function of the macro deformations of composite by the following relations:

$$\langle \sigma^2_{ij} \rangle = \frac{1}{1 - p_0} \lambda^{[2]}_{ijmn} \left(I_{mnk\ell} + c_2 \left(\langle \lambda_{mnk\ell} \rangle - \lambda^*_{mnk\ell} \right) (\lambda^{[3]}_{k\ell pq})^{-1} \right) \langle \varepsilon_{pq} \rangle . \tag{10}$$

Let us assume, that the destruction criterion of the skeleton material is determined by limiting value of intensity of the average tangential stresses in a not destroyed part of the material [4, 6]:

$$I^2_\sigma = \left(\langle \sigma^2_{ij} \rangle' , \langle \sigma^2_{ij} \rangle' \right) = k_2 , \tag{11}$$

where $\langle \sigma^2_{ij} \rangle'$ – deviator of the average over material skeleton stresses, and k_2 – corresponding limiting value, which is a random function of coordinates.

Since it is supposed that the material micro strength is statistical uniform, the one-point distribution function $F(k_2)$ of a random variable k_2 doesn't depend on coordinates and can be described by the exponential-power distribution function in a semi-infinite interval, i.e., by the Weibull distribution [11–16, 22, 23]:

$$F(k_2) = \begin{cases} 0, & k_2 < k_0, \\ 1 - \exp\left(-n(k_2 - k_0)^\alpha\right), & k_2 \geq k_0. \end{cases} \tag{12}$$

Here k_0 is the limiting lower value of the intensity of the tangential stresses k_2 averaged over matrix skeleton, at which the fracture occurs in some macro volumes of matrix, k_1, n and α are the parameters of the distribution function, which can be selected from the condition of the best approximation of the strength spread. These parameters are determined by experimentally for each material.

Let the initial (prior to deformation) micro damage of the matrix be characterized by a porosity p_0. Then, according to the property of ergodicity, the distribution function $F(k_2)$ determines the undamaged fraction of matrix material where the ultimate micro strength is lower than k_2. Therefore, if the stresses in the undamaged part of the matrix are equal to $\left\langle \sigma_{ij}^2 \right\rangle$, then the function $F(I_\sigma^2)$ determines, according to (10)–(12), the fraction of the destroyed micro volumes in the matrix skeleton. The destroyed microvolumes are modeled by pores and then we can write the equation of balance of destroyed micro volumes or porosity [16]:

$$p = p_0 + F(I_\sigma^2)(1 - p_0). \tag{13}$$

In accordance with the given above Eqns. (1)–(10), the matrix skeleton stresses $\left\langle \sigma_{ij}^2 \right\rangle$ can be defined as the function of the composite macro strains $\left\langle \varepsilon_{k\ell} \right\rangle$. Substituting (10) into (13), we obtain a system of equations for determination of the porosity of matrix p, caused by micro destruction, as a function of macro strains:

$$p = p(\left\langle \varepsilon_{k\ell} \right\rangle). \tag{14}$$

Then, substituting p for p_0 into Eqns. (2)–(12), we obtain the nonlinear dependence between the macro stresses $\left\langle \sigma_{ij} \right\rangle$ and the macro strains $\left\langle \varepsilon_{k\ell} \right\rangle$, caused by micro destruction. These relations take into account the strength spread of the material.

3. Effective elastic moduli of composite with orthotropic components. Based on the above model, let us solve the deformation problem for composite which is an isotropic matrix strengthened by ellipsoidal orthotropic inclusions provided that micro damages occur in matrix, i.e., $p_0 \neq 0$. For that the problem on the effective elastic module for such a composite has to be solved and the relations (2) has to be derived in closed species. The solution was presented in [10]. The expressions for determination of effective constants are written down. Passing to matrix designations with super-indices according to the following scheme:

$$11 \to 1, \quad 22 \to 2, \quad 33 \to 3, \quad 23 \to 4, \quad 13 \to 5, \quad 12 \to 6,$$

9 independent constant of the tensor of effective elastic moduli λ_{11}^*, λ_{12}^*, λ_{13}^*, λ_{22}^*, λ_{23}^*, λ_{33}^*, λ_{44}^*, λ_{55}^*, λ_{66}^* of composite under consideration can be calculated according to:

$$\lambda_{ij}^* = \left\langle \lambda_{ij} \right\rangle + c_1 c_2 \lambda_{iq}^{[3]} P_{qk} M_{k\ell} \lambda_{\ell j}^{[3]}, \qquad i, j = 1,2,3, \tag{15}$$

where

$$P_{qk} = \left(\delta_{qk} - M_{q\ell} \lambda'_{\ell k}\right)^{-1},$$

$$\lambda_{ii}^* = \left\langle \lambda_{ii} \right\rangle + c_1 c_2 \lambda_{ii}^{[3]} \frac{4 M_{ii} \lambda_{ii}^{[3]}}{1 - 4 M_{ii} \lambda'_{ii}}, \qquad i = 4,5,6, \tag{16}$$

in here

$$\lambda'_{ij} = c_1 \lambda_{ij}^{[2]} + c_2 \lambda_{ij}^{[1]} - \lambda_{ij}^c, \qquad \lambda_{ij}^{[3]} = \lambda_{ij}^{[1]} - \lambda_{ij}^{[2]},$$

and summation is not performed over repeating indexes.

The components of tensor M_{ij} from (15), (16) are determined in terms of define integrals as follows:

$$M_{11} = \frac{2}{\pi} \int_0^{\pi/2} \left[\lambda_{55}^c \lambda_{66}^c S_1 + \lambda_{44}^c (\lambda_{22}^c S_4 + \lambda_{33}^c S_8) S_3 u_1 + S_5 u_3 + S_6 u_8\right] d\varphi,$$

$$M_{22} = \frac{2}{\pi} \int_0^{\pi/2} \left[\lambda_{44}^c \lambda_{66}^c S_2 + \lambda_{55}^c (\lambda_{11}^c S_3 + \lambda_{33}^c S_9) + S_4 u_2 + S_6 u_9 + S_7 u_3\right] d\varphi,$$

$$M_{33}=\frac{2}{\pi}\int_0^{\pi/2}\left[\lambda_{44}^c\lambda_{55}^c S_{10}+\lambda_{66}^c\left(\lambda_{11}^c S_5+\lambda_{22}^c S_7\right)+S_6u_7+S_8u_2+S_9u_1\right]d\varphi\,,$$

$$M_{23}=\frac{2}{\pi}\int_0^{\pi/2}\left[S_6\left(u_5u_6-\lambda_{11}^c u_4\right)-u_4\left(\lambda_{66}^c S_7+\lambda_{55}^c S_9\right)\right]d\varphi\,,$$

$$M_{13}=\frac{2}{\pi}\int_0^{\pi/2}\left[S_6\left(u_4u_6-\lambda_{22}^c u_5\right)-u_5\left(\lambda_{66}^c S_5+\lambda_{44}^c S_8\right)\right]d\varphi\,,$$

$$M_{12}=\frac{2}{\pi}\int_0^{\pi/2}\left[S_6\left(u_4u_5-\lambda_{33}^c u_6\right)-u_6\left(\lambda_{55}^c S_3+\lambda_{44}^c S_4\right)\right]d\varphi\,,$$

$$M_{44}=\frac{2}{\pi}\int_0^{\pi/2}\left[\lambda_{55}^c\left(\lambda_{11}^c S_5+\lambda_{33}^c S_{10}\right)+\lambda_{66}^c\left(\lambda_{11}^c S_3+\lambda_{22}^c S_2\right)+S_7\left(\lambda_{22}^c\lambda_{55}^c-\right.\right.$$
$$\left.\left.-2\lambda_{23}^c\lambda_{66}^c\right)+S_9\left(\lambda_{33}^c\lambda_{66}^c-2\lambda_{23}^c\lambda_{55}^c\right)+2S_6u_{10}+S_8u_9+S_4u_7\right]d\varphi\,,$$

$$M_{55}=\frac{2}{\pi}\int_0^{\pi/2}\left[\lambda_{44}^c\left(\lambda_{22}^c S_7+\lambda_{33}^c S_{10}\right)+\lambda_{66}^c\left(\lambda_{11}^c S_1+\lambda_{22}^c S_4\right)+S_5\left(\lambda_{11}^c\lambda_{44}^c-\right.\right.$$
$$\left.\left.-2\lambda_{13}^c\lambda_{66}^c\right)+S_8\left(\lambda_{33}^c\lambda_{66}^c-2\lambda_{13}^c\lambda_{44}^c\right)+2S_6u_{11}+S_9u_8+S_3u_7\right]d\varphi\,,$$

$$M_{66}=\frac{2}{\pi}\int_0^{\pi/2}\left[\lambda_{44}^c\left(\lambda_{22}^c S_2+\lambda_{33}^c S_9\right)+\lambda_{55}^c\left(\lambda_{11}^c S_1+\lambda_{33}^c S_8\right)+S_3\left(\lambda_{11}^c\lambda_{44}^c-\right.\right.$$
$$\left.\left.-2\lambda_{12}^c\lambda_{55}^c\right)+S_4\left(\lambda_{22}^c\lambda_{55}^c-2\lambda_{12}^c\lambda_{44}^c\right)+2S_6u_{12}+S_7u_8+S_5u_9\right]d\varphi\,. \quad (17)$$

Here λ_{ij}^c can be represented by a tensor with constant components as follows [10]:

$$\lambda_{ij}^c=\begin{cases}\left\langle\lambda_{ij}\right\rangle, & \text{if } \lambda_{ij}^{[1]}\le\lambda_{ij}^{[2]},\\ \left\langle\lambda_{ij}^{-1}\right\rangle^{-1}, & \text{if } \lambda_{ij}^{[1]}\ge\lambda_{ij}^{[2]},\end{cases}$$

and u_i and S_j are determined from the relations

$$u_1=\lambda_{44}^c\lambda_{66}^c+\lambda_{22}^c\lambda_{55}^c,\quad u_2=\lambda_{55}^c\lambda_{66}^c+\lambda_{11}^c\lambda_{44}^c,\quad u_3=\lambda_{44}^c\lambda_{55}^c+\lambda_{33}^c\lambda_{66}^c,$$
$$u_4=\lambda_{23}^c+\lambda_{44}^c,\quad u_5=\lambda_{13}^c+\lambda_{55}^c,\quad u_6=\lambda_{12}^c+\lambda_{66}^c,$$
$$u_7=\lambda_{11}^c\lambda_{22}^c-{\lambda_{12}^c}^2-2\lambda_{12}^c\lambda_{66}^c,\quad u_8=\lambda_{22}^c\lambda_{33}^c-{\lambda_{23}^c}^2-2\lambda_{23}^c\lambda_{44}^c,$$
$$u_9=\lambda_{11}^c\lambda_{33}^c-{\lambda_{13}^c}^2-2\lambda_{13}^c\lambda_{55}^c,\quad u_{10}=u_5u_6+\lambda_{55}^c\lambda_{66}^c-\lambda_{11}^c\lambda_{23}^c,$$
$$u_{11}=u_4u_6+\lambda_{44}^c\lambda_{66}^c-\lambda_{22}^c\lambda_{13}^c,\quad u_{12}=u_4u_5+\lambda_{44}^c\lambda_{55}^c-\lambda_{33}^c\lambda_{12}^c,$$
$$S_1=A_1\cos^6\varphi,\quad S_2=\frac{A_4-3A_3+3A_2-A_1}{\bar{t}_2^{\,6}}\sin^6\varphi,$$
$$S_3=\frac{A_2-A_1}{\bar{t}_2^{\,2}}\cos^4\varphi\sin^2\varphi,\quad S_4=\frac{A_3-2A_2+A_1}{\bar{t}_2^{\,4}}\cos^2\varphi\sin^4\varphi,$$
$$S_5=\frac{A_2}{\bar{t}_3^{\,2}}\cos^4\varphi,\quad S_6=\frac{A_3-A_2}{\bar{t}_2^{\,2}\bar{t}_3^{\,2}}\cos^2\varphi\sin^2\varphi,$$
$$S_7=\frac{A_4-2A_3+A_2}{\bar{t}_2^{\,4}\bar{t}_3^{\,2}}\sin^4\varphi,\quad S_8=\frac{A_3}{\bar{t}_3^{\,4}}\cos^2\varphi,$$
$$S_9=\frac{A_4-A_3}{\bar{t}_2^{\,2}\bar{t}_3^{\,4}}\sin^2\varphi,\quad S_{10}=\frac{A_4}{\bar{t}_3^{\,6}}. \quad (18)$$

In here

$$\bar{t}_2 = \frac{t_2}{t_1}, \qquad \bar{t}_3 = \frac{t_3}{t_1},$$

where t_1, t_2, t_3 are semi axes of ellipsoids in the directions of axes x_1, x_2, x_3 respectively and the quantities A_j, $j = 1,\dots,4$, take different values, depending on the sign of the parameter $R = h^3 + q^2$. These values will be present bellow. The parameters h and q are determined as follows:

$$h = \frac{3b_1b_3 - b_2^2}{9b_1^2}, \qquad q = \frac{b_2^3}{27b_1^3} - \frac{b_2b_3}{6b_1^2} + \frac{a_1}{2b_1}, \tag{19}$$

where

$$b_1 = \sum_{i=1}^{4}(-1)^i a_i, \qquad b_2 = 3a_1 - 2a_2 + a_3, \qquad b_3 = -3a_1 + a_2, \tag{20}$$

while a_i, $i = 1,\dots,4$, are determined by the formulas:

$$a_1 = \left(\lambda_{55}^c\cos^2\varphi + \frac{\lambda_{44}^c}{\bar{t}_2^2}\sin^2\varphi\right)\left(\lambda_{11}^c\lambda_{66}^c\cos^4\varphi + \frac{\lambda_{22}^c\lambda_{66}^c}{\bar{t}_2^4}\sin^4\varphi + \frac{u_7}{\bar{t}_2^2}\cos^2\varphi\sin^2\varphi\right),$$

$$a_2 = \frac{1}{\bar{t}_3^2}\Bigg[u_3\left(\lambda_{11}^c\cos^4\varphi + \frac{\lambda_{22}^c}{\bar{t}_2^4}\sin^4\varphi\right) - \left((\lambda_{13}^c)^2 + 2\lambda_{13}^c\lambda_{55}^c\right)\Bigg(\lambda_{66}^c\cos^2\varphi +$$

$$+ \frac{\lambda_{22}^c}{\bar{t}_2^2}\sin^2\varphi\Bigg)\cos^2\varphi - \left((\lambda_{23}^c)^2 + 2\lambda_{23}^c\lambda_{44}^c\right)\left(\frac{\lambda_{66}^c}{\bar{t}_2^2}\sin^2\varphi + \lambda_{11}^c\cos^2\varphi\right)\frac{\sin^2\varphi}{\bar{t}_2^2} +$$

$$+ \frac{2\lambda_{44}^c\lambda_{55}^c\lambda_{66}^c + 2u_4u_5u_6 + \lambda_{33}^c u_7}{\bar{t}_2^2}\cos^2\varphi\sin^2\varphi\Bigg],$$

$$a_3 = \frac{1}{\bar{t}_3^4}\left[(\lambda_{33}^c\lambda_{55}^c\lambda_{66}^c + \lambda_{44}^c u_9)\cos^2\varphi + \frac{\lambda_{33}^c\lambda_{44}^c\lambda_{66}^c + \lambda_{55}^c u_8}{\bar{t}_2^2}\sin^2\varphi\right],$$

$$a_4 = \frac{\lambda_{33}\lambda_{44}\lambda_{66}}{\bar{t}_3^6}. \tag{20}$$

At $R = h^3 + q^2 \le 0$, the quantities A_j, $j = 1,\dots,4$, are expressed as follows:

$$A_j = \frac{1}{4b_1|h|}\sum_{i=1}^{3}(-1)^{i+j}\frac{\alpha_i^{j-1}I_i}{d_i}, \qquad j = 1,2,3,$$

$$A_4 = \frac{1}{b_1}\left(1 + \frac{1}{4|h|}\sum_{i=1}^{3}(-1)^i\frac{\alpha_i^3 I_i}{d_i}\right), \tag{21}$$

while

$$I_i = \begin{cases} \dfrac{1}{\sqrt{\alpha_i}}\operatorname{arctg}\sqrt{\alpha_i}, & \alpha_i > 0, \\ -\dfrac{1}{2\sqrt{-\alpha_i}}\ln\dfrac{1+\sqrt{-\alpha_i}}{\left|1-\sqrt{-\alpha_i}\right|}, & \alpha_i < 0, \end{cases} \qquad i = 1,2,3. \tag{22}$$

Here

$$\alpha_1 = \frac{b_2}{3b_1} + 2\operatorname{sign} q\sqrt{|h|}\,v_1, \qquad \alpha_2 = \frac{b_2}{3b_1} - 2\operatorname{sign} q\sqrt{|h|}\,v_2,$$

$$\alpha_3 = \frac{b_2}{3b_1} - 2\operatorname{sign} q\sqrt{|h|}\,v_3, \qquad v_1 = \cos\left(\frac{1}{3}\operatorname{arctg}\frac{\sqrt{-R}}{|h|}\right),$$

$$v_2 = \cos\left(\frac{\pi}{3} - \frac{1}{3}\operatorname{arctg}\frac{\sqrt{-R}}{|h|}\right), \qquad v_3 = \cos\left(\frac{\pi}{3} + \frac{1}{3}\operatorname{arctg}\frac{\sqrt{-R}}{|h|}\right),$$

$$d_1 = (v_1 + v_2)(v_1 + v_3), \quad d_2 = (v_1 + v_2)(v_3 - v_2), \quad d_3 = (v_1 + v_3)(v_3 - v_2), \tag{23}$$

At $R = h^3 + q^2 > 0$, we have

$$A_1 = L(I_1 - (2a - \alpha)I_2 - I_3), \qquad A_2 = L(-\alpha I_1 - z^2 I_2 + \alpha I_3),$$
$$A_3 = L(\alpha^2 I_1 - \alpha z^2 I_2 + (z^2 - 2a\alpha)I_3),$$
$$A_4 = \frac{1}{b_1} - L(\alpha^3 I_1 + z^2(z^2 - 2a\alpha)I_2 + (2az^2 - 4a^2\alpha + z^2\alpha)I_3), \tag{24}$$

where

$$I_1 = \begin{cases} \dfrac{1}{\sqrt{\alpha}}\operatorname{arctg}\sqrt{\alpha}, & \alpha > 0, \\ -\dfrac{1}{2\sqrt{-\alpha}}\ln\dfrac{1 + \sqrt{-\alpha}}{|1 - \sqrt{-\alpha}|}, & \alpha < 0, \end{cases}$$

$$\alpha = \frac{b_2}{3b_1} + 2\operatorname{sign}q\sqrt{|h|}v, \qquad v = \frac{1}{2}\left(r + \frac{1}{r}\right), \qquad r = \frac{\sqrt[3]{|h| + \sqrt{R}}}{|h|^{1/2}},$$

$$I_2 = \frac{1}{2\sqrt{2z}}\left(\frac{1}{2\sqrt{z - a}}\ln\frac{1 + \sqrt{2(z - a)} + z}{1 - \sqrt{2(z - a)} + z} + \frac{1}{\sqrt{z + a}}\operatorname{arc\,ctg}\frac{z - 1}{\sqrt{2(z + a)}}\right),$$

$$I_3 = \frac{1}{2\sqrt{2}}\left(-\frac{1}{2\sqrt{z - a}}\ln\frac{1 + \sqrt{2(z - a)} + z}{1 - \sqrt{2(z - a)} + z} + \frac{1}{\sqrt{z + a}}\operatorname{arc\,ctg}\frac{z - 1}{\sqrt{2(z + a)}}\right), \tag{25}$$

while

$$L = \frac{1}{b_1(\alpha^2 - 2a\alpha + z^2)}, \quad a = \frac{b_2}{3b_1} - \operatorname{sign}q\sqrt{|h|}\,m_1, \quad z = \sqrt{a^2 + 3|h|\,m_2^2}.$$

Meanwhile, m_1 and m_2 take different values, depending on the sign of the parameter h, i.e.

– at $h < 0$:

$$m_1 = \frac{1}{2}\left(r + \frac{1}{r}\right), \qquad m_2 = \frac{1}{2}\left(r - \frac{1}{r}\right), \tag{26}$$

– at $h > 0$:

$$m_1 = \frac{1}{2}\left(r - \frac{1}{r}\right), \qquad m_2 = \frac{1}{2}\left(r + \frac{1}{r}\right). \tag{27}$$

Knowing the tensor M_{ij}, we can use Eqns. (15), (16) to calculate the effective moduli of the orthotropic composite.

The transcendental equation described by (1)–(13) is solved by iterative scheme outlined below.

- The matrix porosity in the n-th approximation $p^{(n)}$ is connected with the limiting value of the intensity of the average tangential stresses over skeleton material in the n-th approximation $k_2^{(n)}$, i.e. with the intensity of the average tangential stresses over skeleton in the n-th approximation $J_\sigma^{2(n)}$. At the same time, the intensity of the average tangential stresses over the matrix skeleton depends according to Eqns. (5)–(10) on the current matrix porosity in the $(n-1)$-th approximation $p^{(n-1)}$, on the effective elastic modules of composite in the $(n-1)$-th approximation $\lambda_{ij}^{*(n-1)}$ and on the macro strains $\langle \varepsilon_{pq} \rangle$. The effective elastic modules are the functions of current matrix porosity in the $(n-1)$-th approximation $p^{(n-1)}$ by Eqns. (2), (4).

Then, based on the relations (11)–(14) we can write the balance equation for porosity in the iterative form as

$$p^{(n-1)} = p_0 + (1 - p_0)F(k_2^{(n-1)}) , \tag{28}$$

where

$$F(k_2^{(n-1)}) = \begin{cases} 0, & k_2^{(n-1)} < k_0, \\ 1 - \exp\left(-n(k_2^{(n-1)} - k_0)^{\alpha}\right), & k_2^{(n-1)} \ge k_0. \end{cases} \tag{29}$$

At the same time, in accordance with the Eqns. (2)–(11) the value of $k_2^{(n-1)}$ in the $(n-1)$-th approximation depends on porosity in the $(n-1)$-th approximation $p^{(n-1)}$ as

$$k_2^{(n-1)} = J_\sigma^{2(n-1)} = J_\sigma^{2(n-1)}\left(\lambda_{ij}^{*(n-1)}, p^{(n-1)}, \langle \varepsilon_{ij} \rangle\right) , \tag{30}$$

and according to the relations (2)–(27) the tensor of effective elastic modules in the $(n-1)$-th approximation depends on porosity in the $(n-1)$-th approximation $p^{(n-1)}$ as

$$\lambda_{ij}^{*(n-1)} = \lambda_{ij}^{*(n-1)}(\lambda_{ij}^{[1]}, \lambda_{ij}^{2}, c_1, p^{(n-1)}, \bar{t}_2, \bar{t}_3) . \tag{31}$$

Hence, the Eqns. (17)–(31), (2) allow us to determine the effective elastic constants of orthotropic composite with porous matrix in terms of macro strains:

$$\lambda_{ij}^{*} = \lim_{n\to\infty} \lambda_{ij}^{*(n)} . \tag{32}$$

Tensor of effective elastic module $\lambda_{ij}^{*(n-1)}$ depends monotonous on porosity $p^{(n-1)}$ therefore the limit exists.

Thus, specifying the macro strains in a material and determining its effective elastic modules from the Eqns. (2), (17)–(31) we can calculate macro stresses.

4. Numerical results. Using the above method and porosity balance equations, let us plot, as an example, a nonlinear macro deformation diagram and analyze the behaviour of the composite with porous isotropic matrix and spheroid orthotropic inclusions. We will consider the case of uniaxial tension of composite:

$$\langle \varepsilon_{11} \rangle \ne 0, \qquad \langle \varepsilon_{22} \rangle = 0.01 ,$$

with elastic constants of inclusions (topaz) [9]:

$$\lambda_{11}^{[1]} = 287\,\text{GPa}, \qquad \lambda_{22}^{[1]} = 365\,\text{GPa}, \qquad \lambda_{33}^{[1]} = 300\,\text{GPa},$$
$$\lambda_{23}^{[1]} = 90\,\text{GPa}, \qquad \lambda_{13}^{[1]} = 85\,\text{GPa}, \qquad \lambda_{12}^{[1]} = 128\,\text{GPa},$$
$$\lambda_{44}^{[1]} = 110\,\text{GPa}, \qquad \lambda_{55}^{[1]} = 135\,\text{GPa}, \qquad \lambda_{66}^{[1]} = 133\,\text{GPa},$$

and elastic modules of the matrix (epoxy):

$$\lambda^{[2]} = 3.7\,\text{GPa}, \qquad \mu^{[2]} = 1.1\,\text{GPa}.$$

The volume concentration of inclusions, initial porosity of the matrix, and the shape of inclusions characterized by parameter $\bar{t}_2$, $\bar{t}_3$ are:

$$c_1 = 0.4 , \qquad p_0 = 0,\ 0.2,\ 0.4 , \qquad \bar{t}_2 = 1 , \qquad \bar{t}_3 = 2 .$$

The parameters characterizing the distribution function of the strength scatter of matrix material are:

$$\alpha = 2 , \qquad n = 10^3,\ 2\times 10^4 , \qquad k_0 = 0.015\,\text{GPa}.$$

Fig. 2–4 show the dependences of $\langle \sigma_{11} \rangle$ on $\langle \varepsilon_{11} \rangle$, $\langle \sigma_{22} \rangle$ on $\langle \varepsilon_{11} \rangle$ and $\langle \sigma_{33} \rangle$ on $\langle \varepsilon_{11} \rangle$ respectively, for various values of p_0 and n, characterizing the distribution function of the strength scatter of the matrix material and without regard of the strength scatter (it is assumed, that variable k_2 in Eq. (11) is constant and equal to k_0).

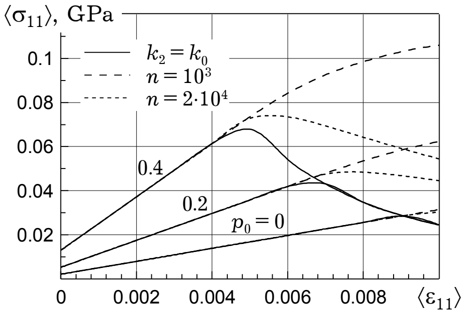

Fig. 2

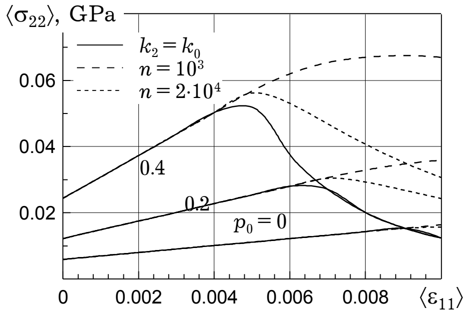

Fig. 3

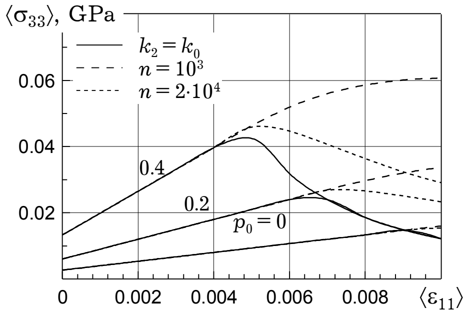

Fig. 4

The solid lines describe the case without regard of the strength scatter of the matrix material, the dotted and dashed lines describe the cases taking into account the strength scatter with parameters $\alpha = 2$, $n = 10^3$ and $\alpha = 2$, $n = 2 \cdot 10^4$, respectively. For various values of matrix porosity, all of the three curves coincide until the onset of the damage. The Figures indicate that the curves corrected for the spread in micro strength, are more smooth, having no breaks. Moreover, by varying the parameters n and α for each material, it is possible to make the theoretical macro deformation curve fit well an experimental one. As it is seen from Figures, the curves plotted regardless of the strength scatter don't display the effect of the initial porosity p_0 on the behaviour of the material after micro cracking sets in. At the same time, the macro deformation diagrams plotted with regard of the strength scatter depend strongly on the initial porosity p_0.

Thus, we may conclude that the proposed numerical-analytical procedure based on the methods of conditional moment function and the iterative secant method allow us to investigate the nonlinear behavior of stochastic anisotropic composites with a porous matrix depending on elastic properties of components, shape and volume concentration of inclusions and matrix porosity.

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ДЕФОРМАЦІЇ ОРТОТРОПНИХ КОМПОЗИТІВ З ОДНОНАПРЯМЛЕНИМИ ЕЛІПСОЇДАЛЬНИМИ ВКЛЮЧЕННЯМИ ПРИ МІКРОПОШКОДЖЕННЯХ МАТРИЦІ

Викладено теорію мікропошкодження матеріалів на основі ортотропної матриці і однонапрямлено орієнтованих волокон, які мають форму триосних еліпсоїдів. Мікроруйнування моделюються порожніми порами. Критерій руйнування в мікрооб'ємі приймається у формі Губера–Мізеса, де границя міцності є випадковою функцією координат із розподілом Вейбулла. Напружено-деформований стан та ефективні властивості матеріалу визначаються з рівнянь теорії пружності для матеріалів на основі ортотропної матриці та однонапрямлено орієнтованих триосних еліпсоїдів. Замикання рівнянь деформування і мікропошкоджуваності здійснюється на основі рівнянь балансу пористості. Побудовано нелінійні залежності сумісних процесів деформування таких матеріалів і мікропошкодження матриці від макродеформацій.

ДЕФОРМАЦИИ ОРТОТРОПНЫХ КОМПОЗИТОВ С ОДНОНАПРАВЛЕННЫМИ ЭЛЛИПСОИДАЛЬНЫМИ ВКЛЮЧЕНИЯМИ ПРИ МИКРОПОВРЕЖДЕНИЯХ МАТРИЦЫ

Изложено теорию микроповреждаемости метериалов на основе ортотропной матрицы и однонаправленных волокон в форме трехосных эллипсоидов. Микроразрушения моделируются пустыми порами. Критерий разрушения в микрообъеме принимается в форме Губера – Мизеса, где предел прочности является случайной функцией координат с распределением Вейбулла. Напряженно-деформированное состояние и эффективные свойства материала с микроповреждениями в компонентах определяются из стохастических уравнений упругости для материалов на основе ортотропной матрицы и однонаправленных трехосных эллипсоидов. Замыкание уравнений деформирования и повреждаемости осуществляется на основании уравнения баланса пористости. Построены нелинейные зависимости совместных процессов деформирования таких материалов и повреждения матрицы от макродеформаций.

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