UDK 539.3

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THERMOELASTICITY WHICH USES FRACTIONAL HEAT CONDUCTION EQUATION

A survey of nonlocal generalizations of the Fourier law and heat conduction equation is presented. More attention is focused on the heat conduction with time and space fractional derivatives and on the theory of thermal stresses based on this equation.

1. Essentials of fractional calculus. In this section we recall the main idea of fractional calculus [5, 7, 23, 37, 38, 45, 51, 54]. It is common knowledge that integrating by parts $n-1$ times the calculation of n-fold primitive of a function $f(t)$ can be reduced to the calculation of a single integral of the convolution type

$$I^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau\,, \qquad (1)$$

where n is a positive constant, $\Gamma(n)$ is the gamma function.

The Riemann – Liouville fractional integral is introduced as a natural generalization of the convolution-type form (1):

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau\,, \qquad \alpha > 0\,. \qquad (2)$$

The Laplace transform rule for the fractional integral (2) reads

$$L\{I^\alpha f(t)\} = \frac{1}{s^\alpha} L\{f(t)\}\,,$$

where s is the transform variable.

The Riemann – Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^α:

$$D_{\mathrm{RL}}^\alpha f(t) = D^n I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,,$$

$$n-1 < \alpha < n\,,$$

and for its Laplace transform requires the knowledge of the initial values of the fractional integral $I^{n-\alpha} f(t)$ and its derivatives of the order $k = 1, 2, \ldots, n-1$:

$$L\{D_{\mathrm{RL}}^\alpha f(t)\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} D^k I^{n-\alpha} f(0^+) s^{n-1-k}, \qquad n-1 < \alpha < n\,.$$

An alternative definition of the fractional derivative was proposed by Caputo [8]:

$$D_C^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} f(\tau)\,d\tau\,, \qquad n-1 < \alpha < n\,.$$

For its Laplace transform rule the Caputo fractional derivative requires the knowledge of the initial values of the function $f(t)$ and its integer derivatives of the order $k = 1, 2, \ldots, n-1$:

$$L\{D_C^\alpha f(t)\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} f^{(k)}(0^+)\, s^{\alpha-1-k}, \qquad n-1 < \alpha < n\,.$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann – Liouville fractional derivative by incorporating the relevant initial conditions [25]. The major utility of the Caputo fractional derivative is caused by the treatment of differential equations of the fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional order), even if the governing equation is of fractional order [41, 54]. If care is taken, the results obtained using the Caputo formulation can be recast to the Riemann – Liouville version and vice versa.

The convolution type form of the Riemann – Liouville fractional integral (2) can be extended to

$$I^{\alpha}_{(a+)}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\zeta)^{\alpha-1} f(\zeta)\,d\zeta\,, \qquad x > a, \qquad \alpha > 0\,, \tag{3}$$

and

$$I^{\alpha}_{(b-)}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\zeta-x)^{\alpha-1} f(\zeta)\,d\zeta, \qquad x < b, \qquad \alpha > 0\,. \tag{4}$$

These integrals are sometimes called the left-sided and right-sided fractional integrals, respectively. It should be mentioned that replacing t by x we have changed notation in equations (3) and (4) in comparison with equation (2) as the following consideration will concern space-fractional differential operators.

The corresponding left-sided and right-sided Riemann – Liouville fractional derivatives of order $\alpha > 0$ are defined by

$$D^{\alpha}_{\mathrm{RL}(a+)}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-\zeta)^{n-\alpha-1} f(\zeta)\,d\zeta\,, \qquad x > a, \quad n-1 < \alpha < n\,,$$

$$D^{\alpha}_{\mathrm{RL}(b-)}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_x^b (\zeta-x)^{n-\alpha-1} f(\zeta)\,d\zeta\,, \qquad x < b, \quad n-1 < \alpha < n\,.$$

The left-sided and right-sided Liouville fractional integrals on the real axis have the form

$$I^{\alpha}_{+}f(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-\zeta)^{\alpha-1} f(\zeta)\,d\zeta, \qquad x \in \mathbb{R}, \qquad \alpha > 0\,,$$

$$I^{\alpha}_{-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (\zeta-x)^{\alpha-1} f(\zeta)\,d\zeta, \qquad x \in \mathbb{R}, \qquad \alpha > 0\,,$$

while the fractional derivatives corresponding to these integrals are expressed by

$$D^{\alpha}_{+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^x (x-\zeta)^{n-\alpha-1} f(\zeta)\,d\zeta\,, \qquad x \in \mathbb{R}, \quad n-1 < \alpha < n\,,$$

$$D^{\alpha}_{-}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_{-\infty}^x (\zeta-x)^{n-\alpha-1} f(\zeta)\,d\zeta\,, \qquad x \in \mathbb{R}, \quad n-1 < \alpha < n\,.$$

The Fourier transform rules for Liouville fractional integrals and derivatives are calculated according to the following formulae:

$$F\{I^{\alpha}_{+}\}f(x) = \frac{1}{(-i\xi)^{\alpha}}F\{f(x)\}\,,$$

$$F\{I^{\alpha}_{-}\}f(x) = \frac{1}{(i\xi)^{\alpha}}F\{f(x)\}, \qquad \alpha > 0\,,$$

$$F\{D_+^\alpha\} f(x) = (-i\xi)^\alpha F\{f(x)\},$$

$$\frac{d^\beta f(x)}{d|x|^\beta} = \frac{\Gamma(1+\beta)}{\pi} \sin\left(\frac{\beta\pi}{2}\right) \int_0^\infty \frac{f(x+\zeta) - 2f(x) + f(x-\zeta)}{\zeta^{1+\beta}} d\zeta, \qquad \alpha > 0,$$

where ξ is the Fourier transform variable and $(\pm i\xi)^\alpha$ means

$$(\pm i\xi)^\alpha = |\xi|^\alpha \exp\left[\pm \frac{1}{2} i\alpha\pi \operatorname{sgn} \xi\right].$$

The Riesz form of the fractional derivative is a symmetric operator with respect to x [22, 42] (we consider this operator for the values $0 < \beta < 2$):

$$\frac{d^\beta f(x)}{d|x|^\beta} = -\frac{1}{\sin(\beta\pi)} \left[\sin\left(\frac{\beta\pi}{2}\right) D_+^\beta f(x) + \sin\left(\frac{\beta\pi}{2}\right) D_-^\beta f(x)\right].$$

The Fourier transform rule for the Riesz derivative reads

$$F\left\{\frac{d^\beta f(x)}{d|x|^\beta}\right\} = -|\xi|^\beta F\{f(x)\}.$$

The Riesz-Feller fractional derivative of order $0 < \beta \le 2$ and skewness ϑ with $\vartheta = \min\{\beta, 2-\beta\}$ modifies the Riesz derivative introducing asymmetry [52]

$$D_\vartheta^\beta f(x) = -\frac{1}{\sin(\beta\pi)} \left\{ \sin\left[\frac{(\beta-\vartheta)\pi}{2}\right] D_+^\beta f(x)(x) + \sin\left[\frac{(\beta+\vartheta)\pi}{2}\right] D_-^\beta f(x) \right\}$$

and has the following Fourier transform rule

$$F\{D_\vartheta^\beta f(x)\} = -|\xi|^\alpha \exp\left[\frac{1}{2} i\vartheta\pi \operatorname{sgn} \xi\right] F\{f(x)\}.$$

The one-dimensional Riesz derivative is the first step in the direction of defining fractional partial operators in higher dimensions. For example, negative powers of the Laplace operator $(-\Delta)^{-\beta/2}$, $\beta > 0$, are called the Riesz potentials (integrals), and their Fourier transforms are defined as [37]

$$F\{(-\Delta)^{-\beta/2} f(\mathbf{x})\} = \frac{1}{|\xi|^\beta} F\{f(\mathbf{x})\}, \qquad \beta > 0,$$

whereas the positive powers $(-\Delta)^{\beta/2}$, $\beta > 0$, are called Riesz derivatives having the Fourier transforms

$$F\{(-\Delta)^{\beta/2} f(\mathbf{x})\} = |\xi|^\beta F\{f(\mathbf{x})\}, \qquad \beta > 0, \tag{5}$$

where ξ is the transform variables vector.

It is obvious that equation (5) is a fractional generalization of the standard formula for the Fourier transform of the Laplace operator

$$F\{-\Delta f(\mathbf{x})\} = |\xi|^2 F\{f(\mathbf{x})\}.$$

It should be noted that the cumbersome aspects of space-fractional differential operators disappear when one computes their integral transforms.

2. Physical backgrounds. The classical theory of heat conduction is based on the Fourier law

$$\mathbf{q} = -k \nabla T \tag{6}$$

relating the heat flux vector q to the temperature gradient, where k is the thermal conductivity of a solid. In combination with the law of conservation of energy, this equation leads to the parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a \Delta T,$$

where a is the thermal diffusivity coefficient.

It should be noted that Eq. (6) is a phenomenological law which states the proportionality of the flux to the gradient of the transported quantity. It is met in several physical contexts with different names. It is well known that from mathematical viewpoint, the Fourier law (6) in the theory of heat conduction, and the Fick law in the theory of diffusion,

$$\mathbf{J} = -\varkappa \nabla c\,,$$

where $\mathbf{J}$ is the matter flux, c is the concentration, $\varkappa$ is the diffusion conductivity, are identical. In combination with the balance equation for mass, the Fick law leads to the classical diffusion equation:

$$\frac{\partial c}{\partial t} = a_c\, \Delta c.$$

Similarly, the classical empirical Darcy law, describing the flow of fluid through a porous medium, states proportionality between the fluid mass flux and the gradient of the pore pressure p

$$\mathbf{J} = -\zeta\, \nabla p$$

and leads to the parabolic equation for the pressure

$$\frac{\partial p}{\partial t} = a_p\, \Delta p.$$

In this paper we discuss heat conduction and thermal stresses, but it obvious that the discussion concerns also diffusion and diffusive stresses as well as the theory of fluid flow through the porous solid.

Nonclassical theories, in which the Fourier law and the standard heat conduction equations are replaced by more general equations, constantly attract the attention of the researchers. The time-nonlocal constitutive equation for the heat flux was considered in [18, 28, 46, 47, 49, 50]

$$\mathbf{q}(t) = -k\int_0^t a(t-\tau)\,\mathrm{grad}\,T(\tau)\,d\tau\,. \tag{7}$$

The corresponding heat conduction equation has the form [47, 48]

$$\frac{\partial T}{\partial t} = a\int_0^t a(t-\tau)\,\Delta T(\tau)\,d\tau\,,$$

while the corresponding thermoelasticity theory was formulated by Chen and Gurtin [17].

The Cattaneo [12, 13] and Vernotte [65] generalization of the Fourier law can be considered as time-nonlocal (7) with «short-tale» exponential kernel and leads to the telegraph equation for the temperature. On its basis Kaliski [36] and Lord and Shulman [39] introduced the theory of generalized elasticity. The first book on the generalized elasticity was published by Podstrigach and Kolyano [2] in 1976.

In the case of constant kernel («full memory») the constitutive equation proposed by Nigmatullin [47] and Green and Naghdi [27] is obtained. In this theory the temperature is described by the wave equation (the ballistic heat conduction):

$$\frac{\partial^2 T}{\partial t^2} = a\,\Delta T\,.$$

Green and Naghdi [27] also considered the theory of thermal stresses based on the wave heat conduction equation. Such a theory is called thermoelasticity without energy dissipation.

For an extensive bibliography on this subject and further generalizations see [1, 2, 14, 15, 30, 31, 33–35, 64] and references therein.

3. Fractional thermoelasticity. In this paper we consider the constitutive equation with the «long-tale» power time-nonlocal kernel (see also [21, 26])

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\,\mathrm{grad}\,T(\tau)\,d\tau, \qquad 0 < \alpha < 1, \tag{8}$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-2}\,\mathrm{grad}\,T(\tau)\,d\tau, \qquad 1 < \alpha < 2. \tag{9}$$

Constitutive equations (8) and (9) can be rewritten in term of fractional integrals and derivatives

$$\mathbf{q}(t) = -kD_{\mathrm{RL}}^{1-\alpha}\,\mathrm{grad}\,T(t), \qquad 0 < \alpha < 1,$$

$$\mathbf{q}(t) = -kI^{\alpha-1}\,\mathrm{grad}\,T(t), \qquad 1 < \alpha < 2,$$

and yield the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta T, \qquad 0 < \alpha < 2, \tag{10}$$

with Caputo fractional derivative of order α. Equation (10) describes the whole spectrum from local heat conduction $\alpha = 0$ through the standard heat conduction $\alpha = 1$ to the ballistic heat conduction $\alpha = 2$.

Space-nonlocal constitutive equations for the heat flux were also considered. In nonlocal thermoelastcity proposed by Eringen [20] (see also [19]), the «short-tale» exponential kernel was assumed. Another type of space-nonlocality is based on «long-tale» kernels resulting in fractional differential operators in space coordinates [16, 24, 52, 53].

The general space-time-fractional heat conduction equation in one-dimensional case has the form [22, 42, 63]:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\frac{\partial^\beta T}{\partial |x|^\beta}, \qquad 0 < \alpha \le 2, \qquad 0 < \beta \le 2,$$

or in the case of higher dimensions [29]

$$\frac{\partial^\alpha T}{\partial t^\alpha} = -a(-\Delta)^{\beta/2}T, \qquad 0 < \alpha \le 2, \qquad 0 < \beta \le 2. \tag{11}$$

Space, time and space-time fractional heat (or diffusion) equations are useful as a mathematical model of many diverse physical phenomena, including amorphous, colloid, glassy and porous materials, fractals, percolation clusters, random and disordered media, comb structures, dielectrics and semiconductors, polymers and biological systems; see [32, 43, 44, 66], among others.

Applications of fractional calculus to various problems of mechanics of solids are reviewed in [3, 4, 6, 9–11, 40, 62]. A quasi-static theory of thermal (or diffusive) stresses based on time-fractional equation (10) was proposed by the author [55], whereas the corresponding theory based on equation (11) was considered in [57].

In the quasi-static theory a thermoelastic state of a solid is governed by the equilibrium equation in terms of displacements

$$\mu\,\boldsymbol{\Delta}\mathbf{u} + (\lambda + \mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} = \beta_T K\,\boldsymbol{\nabla}T,$$

the stress-strain-temperature relation

$$\boldsymbol{\sigma} = 2\mu\,\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_T KT)\,\mathbf{I},$$

the space-time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \gamma\frac{\partial^\alpha\,\mathrm{tr}\,\mathbf{e}}{\partial t^\alpha} = -a(-\Delta)^{\beta/2}T, \qquad 0 < \alpha \le 2, \qquad 0 < \beta \le 2, \tag{12}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{e}$ is the linear strain tensor, T is the temperature, β_T is the thermal coefficient of volumetric expansion, λ and μ are the Lamé constants, $K = \lambda + 2\mu / 3$, $\mathbf{I}$ denotes the unit tensor. The coefficient γ describes the effect of deformation on the thermal state of a solid.

In the framework of the proposed theories, several thermoelastic problems were considered [55, 56, 58, 59, 61] (see also [60]). Because equation (12) in the case $\beta = 2$ and $1 < \alpha < 2$ interpolates the heat conduction equation $\alpha = 1$ and the wave equation $\alpha = 2$, the proposed theory interpolates the classical thermoelasticity and the thermoelasticity without energy dissipation introduced by Green and Naghdi [27].

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ТЕРМОПРУЖНІСТЬ, ЯКА ВИКОРИСТОВУЄ ДРОБОВЕ РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ

Наведено огляд нелокальних узагальнень закону Фур'є і рівняння теплопровідності. Головну увагу звернено на рівняння теплопровідності з похідними за часом і просторовими координатами дробового порядку та на теорію теплових напружень, яка використовує таке рівняння.

ТЕРМОУПРУГОСТЬ, ИСПОЛЬЗУЮЩАЯ ДРОБНОЕ УРАВНЕНИЕ ТЕПЛОПРОВОДНОСТИ

Приведен обзор нелокальных обобщений закона Фурье и уравнения теплопроводности. Основное внимание уделяется уравнению теплопроводности с производными по времени и пространственным координатам дробного порядка и теории тепловых напряжений, использующей такое уравнение.

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Received
18.03.08