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P. I. Kalenyuk[1,2], Z. M. Nytrebych[1], P. Drygaś[2]

METHOD OF SOLVING CAUCHY PROBLEM FOR INHOMOGENEOUS DIFFERENTIAL-OPERATOR EQUATION

We propose a method of solving the Cauchy problem for high order inhomogeneous equation with operator coefficients in a certain linear space. For the right-hand sides of the initial conditions and the equation, which are represented as Stieltjes integrals over a certain measure, the solution of the problem is represented as a sum of Stieltjes integrals over the same measure. We describe some applications of the method for solving the Cauchy problem for inhomogeneous partial differential equations of infinite order in a spatial variable.

1. Statement of the problem. Let $\mathfrak{H}$ be a certain linear space, in which the linear operator A acts with all of its powers A^j defined in $\mathfrak{H}$, $j = 2, 3, \ldots$. Then any vector h from $\mathfrak{H}$ is a C^∞-vector of the operator A [1, p. 66]. Suppose Λ to be an open circle in $\mathbb{C}$ with the centre at point $\lambda = 0$ (if $\Lambda \subseteq \mathbb{R}$, then Λ is a symmetric interval with respect to $\lambda = 0$). Let us denote by $x(\lambda)$ a solution of the equation

$$Ax(\lambda) = \lambda x(\lambda), \qquad \lambda \in \Lambda,$$

considering $x(\lambda)$ to be an eigenvector of the operator A respective to the eigenvalue $\lambda \in \Lambda$, and $x(\lambda) = 0$ when λ is not an eigenvalue of the operator A.

Consider the functions $b_1(\lambda), b_2(\lambda), \ldots, b_n(\lambda)$ analytical in Λ which obviously can be represented as power series

$$b_j(\lambda) = \sum_{k=0}^{\infty} \beta_{jk} \lambda^k,$$

where $\beta_{jk} \in \mathbb{C}$, $k \in \mathbb{N} \cup \{0\}$, $j = 1, \ldots, n$. To these functions, we shall put to a correspondence the following operators:

$$b_j(A) = \sum_{k=0}^{\infty} \beta_{jk} A^k, \qquad j = 1, \ldots, n,$$

whose action in $\mathfrak{H}$ is defined as follows:

$$b_j(A)h = \sum_{k=0}^{\infty} \beta_{jk} A^k h, \qquad j = 1, \ldots, n, \qquad h \in \mathfrak{H},$$

in particular, $b_j(A)x(\lambda) = b_j(\lambda)x(\lambda)$ for $j = 1, \ldots, n$, $\lambda \in \Lambda$.

We shall consider the following Cauchy problem:

$$L\left(\frac{d}{dt}, A\right) U(t) \equiv \frac{d^n U}{dt^n} + \sum_{j=1}^{n} b_j(A) \frac{d^{n-j} U}{dt^{n-j}} = f(t), \tag{1}$$

$$\left. \frac{d^k U}{dt^k} \right|_{t=0} = h_k, \qquad k = 0, 1, \ldots, n-1, \tag{2}$$

where h_k for $k = 0, 1, \ldots, n-1$ are given vectors from the space $\mathfrak{H}$, $f : \mathbb{R} \to \mathfrak{H}$ is a given vector-function, $U : \mathbb{R}_+ \to \mathfrak{H}$ is the sought vector-function.

In the investigations of Cauchy problem for differential-operator equations, a significant place is taken by semigroup theory (see, e.g., [9, 10, 13–15] and their references). Cauchy problem for differential-operator equations has been studied by means of the technique of infinite order operators in the works by Yu. A. Dubinskiy [2, 3] and Ya. V. Radyno [6, 7].

In paper [2], the author has found a representation of the problem solution in integral form by means of the Fourier transform for problem (1), (2), where $A = -i\frac{d}{dx}$ and $\mathfrak{H}$ is a certain subspace $L_2(\mathbb{R})$. To solve the problem (1), (2), where $A = \frac{d}{dx}$ and $\mathfrak{H}$ is a class of entire analytical functions, the differential-symbol method has been used in paper [4]. The problem solution is represented as actions of the differential expressions, whose symbols are right-hand sides of the equations and the initial data, onto certain entire functions of parameters in which the expressions act.

In the present paper, we propose a method of constructing a solution of problem (1), (2) in the form of sum of Stieltjes integrals over a certain measure. That form, in particular, contains the representations of the problem solution obtained in [2] and [4]. Note that the paper proposed is a continuation of [11, 12] to the case of inhomogeneous differential-operator equation.

2. Main results. Let us show the method of solving the problem (1), (2) for the vectors h_k, $k = 0,1,\ldots,n-1$, taken from a special subspace $\mathfrak{H}$ and for $f(t)$ taken from a special class of vector-functions.

Let $\mu(\lambda)$ be a given measure on Λ.

Definition 1. Vector h from $\mathfrak{H}$ is said to belong to $\mathfrak{H}_A \subseteq \mathfrak{H}$, if it could be represented in the form as follows:

$$h = \int_\Lambda R_{\lambda,h}x(\lambda)\,d\mu(\lambda)\,, \tag{3}$$

where $R_{\lambda,h}$ is a linear operator dependent on h and $\lambda \in \Lambda$, which acts in $\mathfrak{H}_A$.

Definition 2. Vector-function $f(t)$ belongs to $N_F(\mathbb{R},\mathfrak{H}_A)$, if $f(t)$ is analytical in $\mathbb{R}$ and for each $t \in \mathbb{R}$ belongs to $\mathfrak{H}_A$ and, besides, there exists a linear analytical in $\mathbb{R}$ operator $F_{\lambda,f}(t)$ dependent on $f(t)$ and $\lambda \in \Lambda$, which for each $t \in \mathbb{R}$ acts in $\mathfrak{H}_A$ and such that

$$f(t) = \int_\Lambda F_{\lambda,f}(t)\,x(\lambda)\,d\mu(\lambda)\,. \tag{4}$$

Hence, each vector-function $f(t)$ from $N_F(\mathbb{R},\mathfrak{H}_A)$ could be represented in a form of Stieltjes integral (4) over the chosen measure with a certain linear operator $F_{\lambda,f}$.

In the differential-operator expression $L\left(\frac{d}{dt}, A\right)$, we shall replace the operator A by the parameter λ and for each $\lambda \in \Lambda$ consider the ordinary differential equation

$$L\left(\frac{d}{dt},\lambda\right)T = 0\,. \tag{5}$$

Denote by

$$T_0(t,\lambda),\ T_1(t,\lambda),\ \ldots,\ T_{n-1}(t,\lambda) \tag{6}$$

the solutions of equation (5) which satisfy the initial conditions

$$\left.\frac{d^kT_j}{dt^k}\right|_{t=0} = \delta_{kj}, \qquad k,j = 0,1,\ldots,n-1\,,$$

where δ_{kj} is a Kronecker symbol.

Lemma 1. *Functions* $T_j(\cdot,\lambda)$, $j=0,1,\ldots,n-1$, *are analytical in* Λ, *and* $T_j(t,\cdot)$, $j=0,1,\ldots,n-1$, *are functions analytical in* $\mathbb{R}$.

P r o o f. By the assumption, functions $b_j(\lambda)$, $j=1,\ldots,n$, are analytical in Λ, so the coefficients of equation (5) are functions analytical in the domain Λ. Let us reduce equation (5) to normal system of first order ordinary differential equations

$$\frac{dX}{dt}=P(\lambda)X, \tag{7}$$

where $X=\operatorname{col}(x_1,x_2,\ldots,x_n)$, $x_1=T$, $x_2=\frac{dT}{dt}$, $\ldots$, $x_n=\frac{d^{n-1}T}{dt^{n-1}}$,

$$P(\lambda)=\begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -b_n(\lambda) & -b_{n-1}(\lambda) & -b_{n-2}(\lambda) & \ldots & -b_2(\lambda) & -b_1(\lambda) \end{Vmatrix}. \tag{8}$$

Let $X_j(t,\lambda)=\operatorname{col}\left(x_{j1}(t,\lambda),x_{j2}(t,\lambda),\ldots,x_{jn}(t,\lambda)\right)$, $j=1,\ldots,n$, be a normal fundamental system of vector-functions of system (7). By the Poincaré theorem [8, p. 59] on analytical dependence of the Cauchy problem solution on the parameter, the vector-functions $X_j(t,\lambda)$, $j=1,\ldots,n$, are analytical in Λ. Since $T_0(t,\lambda)=x_{11}(t,\lambda)$, $T_1(t,\lambda)=x_{21}(t,\lambda)$, $\ldots$, $T_{n-1}(t,\lambda)=x_{n1}(t,\lambda)$, functions (6) are analytical in λ in the domain Λ.

Functions (6), as solutions of ODE (5) with constant (in t) coefficients, are quasipolynomials of t, so those functions are analytical in t variable in $\mathbb{R}$. This completes our proof. ◊

In the differential-operator expression $L\left(\frac{d}{dt},A\right)$, we shall replace the differentiation symbol $\frac{d}{dt}$ by ν, and the operator A by λ. Then we obtain the function $L(\nu,\lambda)$, which is a polynomial of ν and analytical in parameter λ in the domain Λ. Besides, consider the following function:

$$G(\lambda,\nu,t)=\frac{e^{\nu t}-\sum_{j=0}^{n-1}\nu^j T_j(t,\lambda)}{L(\nu,\lambda)}. \tag{9}$$

Lemma 2. *Function of form* (9) *is a solution of the Cauchy problem as follows*:

$$L\left(\frac{d}{dt},\lambda\right)G=e^{\nu t}, \tag{10}$$

$$\left.\frac{d^kG}{dt^k}\right|_{t=0}=0, \qquad k=0,1,\ldots,n-1, \tag{11}$$

and, moreover, $G(\lambda,\cdot,\cdot)$ *is analytical in* Λ, $G(\cdot,\nu,\cdot)$ *and* $G(\cdot,\cdot,t)$ *are functions analytical in* $\mathbb{R}$.

P r o o f. Recall that the set (6) constitutes a normal fundamental system of solutions of equation (5). Let us act by the linear differential expression $L\left(\frac{d}{dt},\lambda\right)$ onto function (9):

$$L\left(\frac{d}{dt},\lambda\right)G = L\left(\frac{d}{dt},\lambda\right)\left\{\left(e^{\nu t} - \sum_{j=0}^{n-1}\nu^j T_j(t,\lambda)\right)L^{-1}(\nu,\lambda)\right\} =$$

$$= L\left(\frac{d}{dt},\lambda\right)\{e^{\nu t}L^{-1}(\nu,\lambda)\} - L^{-1}(\nu,\lambda)\sum_{j=0}^{n-1}\nu^j L\left(\frac{d}{dt},\lambda\right)T_j(t,\lambda) =$$

$$= L^{-1}(\nu,\lambda)L\left(\frac{d}{dt},\lambda\right)e^{\nu t} = L^{-1}(\nu,\lambda)\left\{\frac{d^n}{dt^n} + \sum_{j=1}^{n} b_j(\lambda)\frac{d^{n-j}}{dt^{n-j}}\right\}e^{\nu t} =$$

$$= L^{-1}(\nu,\lambda)\left\{\nu^n + \sum_{j=1}^{n} b_j(\lambda)\nu^{n-j}\right\}e^{\nu t} = L^{-1}(\nu,\lambda)L(\nu,\lambda)e^{\nu t} = e^{\nu t}.$$

Besides, for $k = 0,1,\ldots,n-1$ we have

$$\left.\frac{d^k G}{dt^k}\right|_{t=0} = L^{-1}(\nu,\lambda)\left.\left\{\frac{d^k}{dt^k}\left(e^{\nu t} - \sum_{j=0}^{n-1}\nu^j T_j(t,\lambda)\right)\right\}\right|_{t=0} =$$

$$= L^{-1}(\nu,\lambda)\left.\left\{\nu^k e^{\nu t} - \sum_{j=0}^{n-1}\nu^j \frac{d^k T_j}{dt^k}\right\}\right|_{t=0} = L^{-1}(\nu,\lambda)\left\{\nu^k - \sum_{j=0}^{n-1}\nu^j \delta_{kj}\right\} =$$

$$= L^{-1}(\nu,\lambda)\{\nu^k - \nu^k\} = 0.$$

Since function (9) is a solution of Cauchy problem (10), (11), similarly as in the proof of Lemma 1, one can reduce inhomogeneous differential equation (10) to a system of equations of the following form:

$$\frac{dX}{dt} = P(\lambda)X + \bar{F}, \qquad (12)$$

where $P(\lambda)$ is matrix (8), $\bar{F} = \mathrm{col}\left(0, 0, \ldots, 0, e^{\nu x}\right)$. Function (9), at that, will be the first component of the solution of system (12) satisfying condition $X\big|_{t=0} = 0$. Since the elements of the matrix $P(\lambda)$ are functions analytical in Λ, by Poincaré theorem [8, p. 59], function (9) is analytical in λ parameter in domain Λ.

Note that the function $G(\lambda,\nu,t)$, as a function of ν, is a solution of inhomogeneous equation (10) that contains ν only in the right-hand side $e^{\nu t}$. Therefore, the solution of problem (10), (11) is a quasipolynomial of ν, and so, $G(\cdot,\nu,\cdot)$ is a function analytical in $\mathbb{R}$.

Function (9), as a function of t, is a solution of equation (10) with constant (in t) coefficients with the right-hand side of the form $e^{\nu t}$. Therefore, $G(\cdot,\cdot,t)$ is a quasipolynomial, and so, it is a function analytical in $\mathbb{R}$. This proves our Lemma. ◊

Lemma 3. *If* $f \in N_F(\mathbb{R},\mathfrak{H}_A)$ *then there holds the equality as follows*:

$$F_{\lambda,f}\left(\frac{d}{d\nu}\right)\{e^{\nu t}x(\lambda)\} = e^{\nu t}F_{\lambda,f}(t)\,x(\lambda), \qquad (t,\lambda) \in \mathbb{R}\times\Lambda. \qquad (13)$$

P r o o f. Let us develop $F_{\lambda,f}(t)$ as a series:

$$F_{\lambda,f}(t) = \sum_{n=0}^{\infty} c_{\lambda,f,n}t^n.$$

Then we have

$$F_{\lambda,f}\left(\frac{d}{d\nu}\right)\{e^{\nu t}x(\lambda)\} = \sum_{n=0}^{\infty} c_{\lambda,f,n}\,\frac{d^n}{d\nu^n}\{e^{\nu t}x(\lambda)\} =$$

$$= \sum_{n=0}^{\infty} c_{\lambda,f,n}\{t^n e^{\nu t}x(\lambda)\} = e^{\nu t}\left(\sum_{n=0}^{\infty} c_{\lambda,f,n}t^n\right)x(\lambda) = e^{\nu t}F_{\lambda,f}(t)\,x(\lambda).$$

The proof is complete. ◊

Lemma 4. *Let* $\chi(t,\lambda)$ *be an arbitrary function analytical in* $\mathbb{R}\times\Lambda$, *and let the operator* A *commute with* $\frac{d}{dt}$. *Then there holds the equality as follows:*

$$L\left(\frac{d}{dt}, A\right)\{\chi(t,\lambda)x(\lambda)\} = \left\{L\left(\frac{d}{dt},\lambda\right)\chi(t,\lambda)\right\}x(\lambda), \qquad (t,\lambda)\in\mathbb{R}\times\Lambda. \quad (14)$$

P r o o f. First of all, note that if $x(\lambda)$ is not an eigenvector of the operator A then $x(\lambda)=0$ and equality (14) moves to an identity. If $x(\lambda)$ is an eigenvector of the operator A, $\lambda\in\Lambda$, then the proof is similar to the proof of Lemma 1 in [12]. The proof is complete. ◊

Corollary. *Let the functions system* (6) *be a normal fundamental system of solutions of equation* (5), $G(\lambda,\nu,t)$ *be function* (9), *and let the operator* A *commute with* $\frac{d}{dt}$. *Then the following equalities hold:*

$$L\left(\frac{d}{dt}, A\right)\{T_k(t,\lambda)x(\lambda)\} = 0, \qquad k = 0,1,\ldots,n-1, \quad (15)$$

$$L\left(\frac{d}{dt}, A\right)\{G(\lambda,\nu,t)x(\lambda)\} = e^{\nu t}x(\lambda), \qquad \lambda\in\Lambda. \quad (16)$$

P r o o f. Equalities (15) and (16) follow from (14), if one takes $T_k(t,\lambda)$ and $G(\lambda,\nu,t)$ respectively as $\chi(t,\lambda)$ and makes use of equalities (5) and (10). The proof is complete. ◊

Now we pass on to constructing a solution of problem (1), (2). Suppose in the initial conditions (2) $h_k\in\mathfrak{H}_A$, $k=0,1,\ldots,n-1$. This means that there exist linear operators R_{λ,h_k} such that

$$h_k = \int_\Lambda R_{\lambda,h_k}x(\lambda)\,d\mu(\lambda), \qquad k = 0,1,\ldots,n-1. \quad (17)$$

Let in equation (1) $f\in N_F(\mathbb{R},\mathfrak{H}_A)$ and, besides, suppose the conditions (**A**) and (**B**) to be fulfilled, where

(**A**) is a condition of existence of such Stieltjes integrals:

$$\int_\Lambda \left[F_{\lambda,f}\left(\frac{d}{d\nu}\right)\{G(\lambda,\nu,t)\,x(\lambda)\}\right]\Bigg|_{\nu=0} d\mu(\lambda),$$

$$\int_\Lambda R_{\lambda,h_k}\{T_k(t,\lambda)x(\lambda)\}\,d\mu(\lambda), \qquad k = 0,1,\ldots,n-1;$$

(**B**) is a condition of fulfillment of the following equalities:

$$L\left(\frac{d}{dt}, A\right)\int_\Lambda R_{\lambda,h_k}\{T_k(t,\lambda)x(\lambda)\}d\mu(\lambda) =$$

$$= \int_\Lambda R_{\lambda,h_k}\left[L\left(\frac{d}{dt}, A\right)\{T_k(t,\lambda)x(\lambda)\}\right]d\mu(\lambda), \qquad k = 0,1,\ldots,n-1,$$

$$L\left(\frac{d}{dt}, A\right)\int_\Lambda \left[F_{\lambda,f}\left(\frac{d}{d\nu}\right)\{G(\lambda,\nu,t)x(\lambda)\}\right]\Bigg|_{\nu=0} d\mu(\lambda) =$$

$$= \int_\Lambda \left[F_{\lambda,f}\left(\frac{d}{d\nu}\right)L\left(\frac{d}{dt}, A\right)\{G(\lambda,\nu,t)x(\lambda)\}\right]\Bigg|_{\nu=0} d\mu(\lambda).$$

Theorem 1. *Let, in conditions* (2), $h_k \in \mathfrak{H}_A$ *for each* $k = 0,1,\dots,n-1$, *i. e. equalities* (17) *hold, besides, in equation* (1), $f(t)$ *belong to* $N_F(\mathbb{R},\mathfrak{H}_A)$ *and be represented in the form* (4), *the linear operator* A *act in* $\mathfrak{H}_A$ *and commute with* $\frac{d}{dt}$, *and conditions* (**A**), (**B**) *be fulfilled. Then the solution of problem* (1), (2) *could be expressed in the form as follows:*

$$U(t) = \sum_{k=0}^{n-1}\int_\Lambda R_{\lambda,h_k}\{T_k(t,\lambda)x(\lambda)\}d\mu(\lambda) +$$

$$+ \int_\Lambda \left[F_{\lambda,f}\left(\frac{d}{d\nu}\right)\{G(\lambda,\nu,t)x(\lambda)\}\right]\Bigg|_{\nu=0} d\mu(\lambda). \tag{18}$$

P r o o f. Let us show that under the assumptions made, vector-function (18) satisfies equation (1). In fact, by the conditions (**A**) and (**B**), we have

$$L\left(\frac{d}{dt}, A\right)U(t) = \sum_{k=0}^{n-1}\int_\Lambda R_{\lambda,h_k}\left[L\left(\frac{d}{dt}, A\right)\{T_k(t,\lambda)x(\lambda)\}\right]d\mu(\lambda) +$$

$$+ \int_\Lambda \left[F_{\lambda,f}\left(\frac{d}{d\nu}\right)L\left(\frac{d}{dt}, A\right)\{G(\lambda,\nu,t)x(\lambda)\}\right]\Bigg|_{\nu=0} d\mu(\lambda).$$

From equalities (15) and (16), we obtain

$$L\left(\frac{d}{dt}, A\right)U(t) = \sum_{k=0}^{n-1}\int_\Lambda R_{\lambda,h_k}\{0\}d\mu(\lambda) + \int_\Lambda \left[F_{\lambda,f}\left(\frac{d}{d\nu}\right)\{e^{\nu t}x(\lambda)\}\right]\Bigg|_{\nu=0} d\mu(\lambda).$$

The first n terms in the last sum are equal to zero by the linearity of the operators R_{λ,h_k}, $k = 0,1,\dots,n-1$, and the last term, by Lemma 3, equals to $\int_\Lambda \left[e^{\nu t}F_{\lambda,f}(t)x(\lambda)\right]\Big|_{\nu=0} d\mu(\lambda)$. Therefore, we have

$$L\left(\frac{d}{dt}, A\right)U(t) = \int_\Lambda F_{\lambda,f}(t)\,x(\lambda)\,d\mu(\lambda).$$

Taking into account equality (4), we obtain $L\left(\frac{d}{dt}, A\right)U(t) = f(t)$.

Now we shall prove the fulfillment of conditions (2). For $j = 0,1,\dots,n-1$, we have

$$\left.\frac{d^jU}{dt^j}\right|_{t=0} = \sum_{k=0}^{n-1}\int_\Lambda R_{\lambda,h_k}\left\{\frac{d^jT_k}{dt^j}x(\lambda)\right\}\Bigg|_{t=0} d\mu(\lambda) +$$

$$+ \int_\Lambda \left[F_{\lambda,f}\left(\frac{d}{d\nu}\right)\left\{\frac{d^jG}{dt^j}x(\lambda)\right\}\Bigg|_{t=0}\right]\Bigg|_{\nu=0} d\mu(\lambda).$$

Considering (11) and the fact that $\left.\frac{d^jT_k}{dt^j}\right|_{t=0} = \delta_{jk}$, we obtain

$$\left.\frac{d^j U}{dt^j}\right|_{t=0} = \sum_{k=0}^{n-1} \int_{\Lambda} R_{\lambda,h_k} \{\delta_{kj} x(\lambda)\} d\mu(\lambda) = \int_{\Lambda} R_{\lambda,h_j} x(\lambda)\, d\mu(\lambda).$$

By equalities (17), we have $\left.\frac{d^j U}{dt^j}\right|_{t=0} = h_j$, where $j = 0, 1, \ldots, n-1$. This proves our theorem. ◊

Now we shall give examples of the operators A and the respective spaces $\mathfrak{H}$ and $\mathfrak{H}_A$, when the conditions of Theorem 1 are fulfilled.

Example 1. Let $\mathfrak{H} = L_2(\mathbb{R})$, $A = -i\frac{d}{dx}$, $i^2 = -1$, $\Lambda = \mathbb{R}$, $\mathfrak{H}_A = H^{\infty}(\Lambda)$. The space $\mathfrak{H}_A$ consists of such functions $h(x)$ that the Fourier transform $\widehat{h}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\Lambda} h(x) e^{-ix\lambda} dx$ is finite in Λ. Problem (1), (2) in this case will have the form

$$L\left(\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}\right) U(t,x) \equiv \frac{\partial^n U}{\partial t^n} + \sum_{j=1}^{n} b_j \left(-i\frac{\partial}{\partial x}\right) \frac{\partial^{n-j} U}{\partial t^{n-j}} = f(t,x), \tag{19}$$

$$\frac{\partial^k U}{\partial t^k}(0,x) = h_k(x), \qquad k = 0, 1, \ldots, n-1. \tag{20}$$

The eigenvector $x(\lambda)$ of the operator A is $e^{i\lambda x}$. As measure $\mu(\lambda)$ we take the Lebesgue measure, i.e. $d\mu(\lambda) = d\lambda$. For any function $h(x)$ from $H^{\infty}(\mathbb{R})$, we have the representation $h(x) = \int_{\mathbb{R}} R_{\lambda,h} e^{i\lambda x} d\lambda$, where $R_{\lambda,h} = \frac{1}{\sqrt{2\pi}} \widehat{h}(\lambda)$.

The class $N_F(\mathbb{R}, \mathfrak{H}_A)$ for problem (19), (20) is the set of all functions $f(t,x)$ analytical in $\mathbb{R}$ in t variable, which for fixed $t \in \mathbb{R}$ belong to $H^{\infty}(\mathbb{R})$. Then $f(t,x) = \int_{\mathbb{R}} F_{\lambda,f}(t) e^{i\lambda x} d\lambda$, where $F_{\lambda,f}(t) = \frac{1}{\sqrt{2\pi}} \widehat{f}(t,\lambda)$, $\widehat{f}(t,\lambda)$ is a Fourier transform of the function $f(t,x)$ in x variable.

The operator $A = -i\frac{d}{dx}$ commutes with $\frac{d}{dt}$, the condition (**A**) of existence of the integrals

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\widehat{f}\left(\frac{d}{d\nu}, \lambda\right) \{G(\lambda, \nu, t) e^{ix\lambda}\} \right]\Big|_{\nu=0} d\lambda,$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{h}_k(\lambda)\, T_k(t,\lambda)\, e^{ix\lambda} d\lambda, \; k = 0, 1, \ldots, n-1,$$

holds by the finiteness of $\widehat{h}_k(\lambda)$, $k = 0, 1, \ldots, n-1$, and $\widehat{f}\left(\frac{d}{d\nu}, \lambda\right)$ in Λ. The action of any differential expression $\widehat{f}\left(\frac{d}{d\nu}, \lambda\right)$ onto $G(\lambda, \nu, t) e^{ix\lambda}$ with respect to the parameter ν is correctly defined since the function $G(\cdot, \nu, \cdot)$ is an entire analytical function of first order (see [5, p. 314]). The condition (**B**) holds as well. The operators $b_j\left(-i\frac{\partial}{\partial x}\right)$, $j = 1, \ldots, n$, act invariantly in $H^{\infty}(\mathbb{R})$.

By Theorem 1, we obtain such a result as to the solvability of problem (19), (20).

Theorem 2. *Let for each* $k = 0,1,\ldots,n-1$ *the functions* $h_k(x)$ *belong to* $H^{\infty}(\mathbb{R})$, $f(t,\cdot)$ *be a function analytical in* $\mathbb{R}$, *and* $f(\cdot,x) \in H^{\infty}(\mathbb{R})$. *Then the solution of problem* (19), (20) *could be expressed in the form as follows:*

$$U(t,x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n-1} \int_{\mathbb{R}} \widehat{h}_k(\lambda)\, T_k(t,\lambda) e^{ix\lambda} d\lambda +$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\widehat{f}\left(\frac{d}{d\nu},\lambda\right) \{G(\lambda,\nu,t)\, e^{ix\lambda}\} \right]\Big|_{\nu=0} d\lambda .$$

Example 2. Let in equation (1) $A = \dfrac{d}{dx}$, $\mathfrak{H} = \mathfrak{A}$ be the class of functions $h(x)$ analytical in $\mathbb{R}$, $\Lambda = \mathbb{R}$, $e^{\lambda x}$ be an eigenvector of the operator A. Problem (1), (2) is a Cauchy problem for the equation

$$L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right) U(t,x) \equiv \frac{\partial^n U}{\partial t^n} + \sum_{j=1}^{n} b_j\left(\frac{\partial}{\partial x}\right)\frac{\partial^{n-j} U}{\partial t^{n-j}} = f(t,x) \qquad (21)$$

with initial conditions (20).

As a measure $\mu(\lambda)$, we take the Dirac measure, i.e. $d\mu(\lambda) = \delta(\lambda)\, d\lambda$. As $\mathfrak{H}_A = \mathfrak{A}_p$, we take the class of functions analytical in $\mathbb{R}$ with the growth order not greater than $p \in \mathbb{R}_+$ (this order is assigned by the behavior of the symbols $b_j(\lambda)$, $j = 1,\ldots,n$, see [4, p. 122]). Then each function $h(x)$ from $\mathfrak{A}_p$, as an analytical function in $\mathbb{R}$, could be represented in the form

$$h(x) = \int_{\mathbb{R}} R_{\lambda,h} e^{\lambda x} \delta(\lambda)\, d\lambda ,$$

or

$$h(x) = R_{\lambda,h} e^{\lambda x}\Big|_{\lambda=0} ,$$

where $R_{\lambda,h} = h\left(\dfrac{d}{d\lambda}\right)$, i.e. $R_{\lambda,h} = \displaystyle\sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}\left(\frac{d}{d\lambda}\right)^k$.

As $N_F(\mathbb{R},\mathfrak{H}_A)$, we take the class of functions $f(t,x)$ analytical in $\mathbb{R}^2$, such that $f(\cdot,x)$ belongs to $\mathfrak{A}_p$. Then

$$f(t,x) = F_{\lambda,f}(t)\, e^{\lambda x}\Big|_{\lambda=0} ,$$

where $F_{\lambda,f}(t) = f\left(t,\dfrac{d}{d\lambda}\right)$, i.e. $F_{\lambda,f}(t) = \displaystyle\sum_{k=0}^{\infty} \frac{\frac{\partial^k f}{\partial x^k}(t,0)}{k!}\left(\frac{d}{d\lambda}\right)^k$.

In this case, the operator $A = \dfrac{d}{dx}$ commutes with $\dfrac{d}{dt}$, the existence of Stieltjes integrals in condition (**A**) at the expense of Dirac measure is reduced to the convergence of such series:

$$f\left(\frac{\partial}{\partial \nu},\frac{\partial}{\partial \lambda}\right)\{G(\lambda,\nu,t)\, e^{\lambda x}\}\Big|_{\lambda=0,\nu=0} ,$$

$$h_k\left(\frac{\partial}{\partial \lambda}\right)\{T_k(t,\nu)\, e^{\lambda x}\}\Big|_{\lambda=0} , \qquad k = 0,1,\ldots,n-1 .$$

Those integrals converge at the expense of choosing the classes $N_F(\mathbb{R},\mathfrak{A}_p)$ and $\mathfrak{A}_p$. The condition (**B**) gets the form

$$L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\left[h_k\left(\frac{\partial}{\partial\lambda}\right)\{T_k(t,\lambda)e^{\lambda x}\}\Big|_{\lambda=0}\right]=$$

$$=h_k\left(\frac{\partial}{\partial\lambda}\right)\left[L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\{T_k(t,\lambda)e^{\lambda x}\}\right]\Big|_{\lambda=0},$$

$$L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\left[f\left(\frac{\partial}{\partial\nu},\frac{\partial}{\partial\lambda}\right)\{G(\lambda,\nu,t)e^{\lambda x}\}\Big|_{\lambda=0,\nu=0}\right]=$$

$$=\left[f\left(\frac{\partial}{\partial\nu},\frac{\partial}{\partial\lambda}\right)L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\{G(\lambda,\nu,t)e^{\lambda x}\}\right]\Big|_{\lambda=0,\nu=0}.$$

Those equalities hold by the analyticity of the respective functions in the parameters λ and ν.

By Theorem 1, we can formulate the result as follows.

Theorem 3. *Let for each* $k=0,1,\ldots,n-1$ *the functions* $h_k(x)$ *belong to* $\mathfrak{A}_p$ *and* $f\in N_F(\mathbb{R},\mathfrak{A}_p)$. *Then the solution of problem* (21), (20) *could be expressed in the form*

$$U(t,x)=f\left(\frac{\partial}{\partial\nu},\frac{\partial}{\partial\lambda}\right)\{G(\lambda,\nu,t)\,e^{\lambda x}\}\Big|_{\lambda=0,\nu=0}+$$

$$+\sum_{k=0}^{n-1}h_k\left(\frac{\partial}{\partial\lambda}\right)\{T_k(t,\lambda)e^{\lambda x}\}\Big|_{\lambda=0}.$$

3. Conclusions. In the present paper, we propose a method of solving a Cauchy problem for inhomogeneous differential-operator equation of order n. In a special class of vector-functions, the problem solution is represented as a sum of Stieltjes integrals over a certain measure. Such a representation includes, as particular cases, an integral representation of the Cauchy problem solution for PDE obtained by means of the Fourier transform, as well as a representation of the Cauchy problem solution for PDE of generally infinite order in spatial variable obtained by means of the differential-symbol method.

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МЕТОД РОЗВ'ЯЗУВАННЯ ЗАДАЧІ КОШІ ДЛЯ НЕОДНОРІДНОГО ДИФЕРЕНЦІАЛЬНО-ОПЕРАТОРНОГО РІВНЯННЯ

Запропоновано метод розв'язування задачі Коші для неоднорідного рівняння високого порядку з операторними коефіцієнтами у деякому лінійному просторі. Для правих частин початкових умов та рівняння, які зображаються як інтеграли Стілтьєса за деякою мірою, розв'язок задачі зображено у вигляді суми інтегралів Стілтьєса за цією ж мірою. Подано приклади застосування методу до розв'язування задачі Коші для неоднорідних диференціальних рівнянь із частинними похідними нескінченного порядку за просторовою змінною.

МЕТОД РЕШЕНИЯ ЗАДАЧИ КОШИ ДЛЯ НЕОДНОРОДНОГО ДИФФЕРЕНЦИАЛЬНО-ОПЕРАТОРНОГО УРАВНЕНИЯ

Предложен метод решения задачи Коши для неоднородного уравнения высокого порядка с операторными коэффициентами в некотором линейном пространстве. Для правых частей начальных условий и уравнения, которые представляются в виде интегралов Стилтьеса по некоторой мере, решение задачи представлено в виде суммы интегралов Стилтьеса по этой же мере. Приведены примеры применения метода к решению задачи Коши для дифференциальных уравнений в частных производных бесконечного порядка по пространственной переменной.

[1] L'viv Polytechnic Nat. Univ., L'viv,
[2] Univ. of Rzeszów, Rzeszów, Poland

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