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BOUNDEDNESS OF WEAK SOLUTIONS OF NONDIAGONAL SINGULAR PARABOLIC SYSTEM EQUATIONS

The boundedness of weak solutions of a nondiagonal parabolic system of singular quasilinear differential equations with matrix of coefficients, satisfying to special structure conditions, is studied. Thus, the technique, basing on estimating the linear combinations of unknowns, is employed.

1. Introduction. As it is well–known, the De Giorgi–Nash–Moser estimates in the elliptic system case, in general, are no valid. The example of an unbounded solution of a linear elliptic system with bounded coefficients was built up by De Giorgi in [3]. Due to J. Nečas and J. Souček, there is also another example of a nonlinear elliptic system with sufficiently smooth coefficients, which the weak solution not belongs to $W^{2,2}$. Since the elliptic system can be interpretive as a special case of the parabolic version, these examples concern also to the parabolic case.

Using in this work the technical equipment has been earlier applied to the case of weakly nonlinear systems in [6] (see also [4, 7] and [5]). The main idea of our approach is as follows: we receive estimations of solution u^μ, $\mu = 1, \dots, N$ from estimations of their certain linear combinations

$$\begin{gathered} H^1 = \alpha^{11}u^1 + \dots + \alpha^{N1}u^N, \\ \dots\dots\dots \\ H^N = \alpha^{1N}u^1 + \dots + \alpha^{NN}u^N, \end{gathered} \tag{1}$$

or, in general, of certain functions $H(t, x, u^1, \dots, u^N)$.

2. Basic notations and hypotheses. We shall be concerned of the following system of N equations

$$\begin{aligned} & u_t^1 - \frac{\partial}{\partial x_i}\left(A_i^{(1)}(x, u^1, \dots, u^N, u_x^1, \dots, u_x^N)\right) = \\ & \qquad\qquad = B^{(1)}(x, u^1, \dots, u^N, u_x^1, \dots, u_x^N), \\ & \qquad \dots\dots\dots\dots\dots\dots \\ & u_t^N - \frac{\partial}{\partial x_i}\left(A_i^{(N)}(x, u^1, \dots, u^N, u_x^1, \dots, u_x^N)\right) = \\ & \qquad\qquad = B^{(N)}(x, u^1, \dots, u^N, u_x^1, \dots, u_x^N), \qquad x \in Q. \end{aligned} \tag{2}$$

with the Dirichlet type boundary conditions of the form

$$\begin{gathered} (u^\mu - g^\mu)(x,t) \in W_0^{1,p}(\Omega) \qquad \text{a.e.} \quad t \in (0,T), \\ u^\mu(x,0) = u_0^\mu(x). \end{gathered} \tag{3}$$

We assume that $(u^1, \dots, u^N) \in C(0,T; L^2(\Omega)) \bigcap L^p(0,T; W^{1,p}(\Omega))$ and a solution of the system (2) with Dirichlet data (3) we understand in the weak sense, as in [2].

Definition. Let Ω be a domain in $\mathbb{R}^n$ $(n \in \mathbb{N})$, $\partial\Omega$ be a path of its boundary and $W(\Omega)$ be a Sobolev space in Ω. By $W(\partial\Omega)$ we denote the space of functions u defined on $\partial\Omega$ with the finite norm $\|u\|_{W(\partial\Omega)} = \inf\limits_{\psi} \|\psi\|_{W(\Omega)}$, where the infimum is taken by all functions $\psi \in W(\Omega)$ such that $\psi = u$ a. e. on $\partial\Omega$.

Let Ω is a bounded domain in $\mathbb{R}^n$. We use the following notations: $Q = (0,T] \times \Omega$; $S = \partial\Omega \times (0,T]$; $\partial Q \equiv \{\Omega \times \{0\}\} \bigcup \{\partial\Omega \times (0,T]\}$; $x \in \Omega$; $t \in (0,T]$, $(T > 0)$; $1 < p < 2$; $p < n$; $i,j = 1,...,n$; $\mu,\nu = 1,...,N$; $u,v,w \in C(0,T;L^2(\Omega)) \bigcap L^p(0,T;W^{1,p}(\Omega))$; $W_0^{1,p}(\Omega)$ is the subspace of functions in $W^{1,p}(\Omega)$ vanishing on $\partial\Omega$ (in the sense of traces for a.e $t \in (0,T]$). Throughout in the paper, we briefly denote by $|s|$ and $|s_i|$ the distance in (Nn)–dimensional and n–dimensional Euclidean space respectively, i.e

$$|s| = \sqrt{\sum_{\nu=1}^{N}\sum_{i=1}^{n} (s_i^\nu)^2}, \quad |s_i| = \sqrt{\sum_{i=1}^{n} (s_i^\nu)^2}, \tag{4}$$

where s and s_i denote some (Nn)–component and n–component vector respectively.

The parabolicity property of system (2) means that its part without derivatives by time is elliptic. Thus, inheriting to [1], the ellipticity property of a system of differential equations is understood in the following sense:

$$\exists\lambda > 0,\ 0 < F = F(x) \in L^{p/(p-1)}(Q) \mid \quad \forall s_i^\mu \in \mathbb{R}^{Nn},\ \forall r^\mu \in \mathbb{R}^N,\ \forall x \in \mathbb{R}^n;$$
$$\sum_{i=1}^{n}\sum_{\nu=1}^{N} A_i^\nu(x,r,s) s_i^\nu \geq \lambda |s|^p - F. \tag{5}$$

Concerning $A_i^\mu(x,r,s)$, we assume that these are measurable functions of the form $\Omega \times \mathbb{R}^3 \times \mathbb{R}^{3n} \longrightarrow \mathbb{R}$, satisfying the ellipticity condition and the following growth conditions: $\exists\ \Lambda_2 > 0 \mid \ \forall s_i^\mu \in \mathbb{R}^{Nn},\ \forall r^\mu \in \mathbb{R}^N,\ \forall x \in \mathbb{R}^n,\ i = 1,...,n;$

$$|A_i^\mu(x,r,s)| \leq \Lambda_2 |s|^{p-1}, \tag{6}$$

and the following structure conditions:

$$\begin{gathered}
\exists\ \alpha^{\mu\nu} \in \mathbb{R}^{NN}; \quad \mathrm{Det}\begin{vmatrix} \alpha^{11} & \cdots & \alpha^{1N} \\ \vdots & \ddots & \vdots \\ \alpha^{N1} & \cdots & \alpha^{NN} \end{vmatrix} \neq 0 \text{ such that} \\
\forall s_i^\mu \in \mathbb{R}^{Nn}, \quad \forall r^\mu \in \mathbb{R}^N, \quad \forall x \in \mathbb{R}^n;\ \mu = 1,...,N; \\
\left| \sum_{\nu=1}^{N} \alpha^{\mu\nu} A_i^{(\nu)}(x,r,s) - \lambda^\mu(x,r,s) \left(\sum_{\nu=1}^{N} \alpha^{\mu\nu} s_i^\nu \right) \right| \leq \\
\leq \eta^\mu(x,r,s) \left| \sum_{\nu=1}^{N} \alpha^{\mu\nu} s_i^\nu \right| + \xi^\mu(x,r,s) + F^\mu,
\end{gathered} \tag{7}$$

where $\lambda^\mu = \lambda^\mu(x,r,s) > 0$, $\eta^\mu = \eta^\mu(x,r,s) > 0$, $\xi^\mu = \xi^\mu(x,r,s) > 0$ are some measurable functions of x, u^μ, $u_{x_i}^\mu$ of the form $\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}$, satisfying the following growth conditions:

$$\exists\ \Lambda_1, \Lambda_2 > 0 | \quad \forall s_i^\mu \in \mathbb{R}^{Nn}, \quad \forall r^\mu \in \mathbb{R}^\mu, \quad \forall x \in \mathbb{R}^n;\ \mu = 1,...,N;$$

$$0 < \Lambda_1 \left| \sum_{\nu=1}^{N} \alpha^{\mu\nu} s_i^\nu \right|^{p-2} \leq \lambda^\mu(x,r,s) \leq \Lambda_2 \left| \sum_{\nu=1}^{N} \alpha^{\mu\nu} s_i^\nu \right|^{p-2}; \tag{8}$$

$$\xi^\mu(x,r,s) \leq \xi_0 |s|^\omega, \quad 0 < \omega = \frac{p(p-1)(1-\kappa_1)}{(n+p)}; \quad \xi_0 > 0, \tag{9}$$

$$F^\mu \in L^\sigma(Q), \quad \sigma = \frac{(p+n)}{(p-1)(1-\kappa_1)}, \quad \kappa_1 \in (0,1), \tag{10}$$

$$\alpha^{\mu\mu} > 1; \tag{11}$$

$$\alpha^{\mu\nu} < 1 \quad \text{for} \quad \mu \neq \nu; \tag{12}$$

$$N \max[1/p, \Lambda_2] \max[(\alpha^{11})^{-1}, ..., (\alpha^{NN})^{-1}] \leq \Lambda_1/(2^p p); \tag{13}$$

$$3N \max[\eta_0, \xi_0] \leq \Lambda_1/(2^{p+1}p). \tag{14}$$

Remark. It is uneasy for checking up that, as $F^\mu \in L^{\frac{(p+n)}{(p-1)(1-\kappa_1)}}$, the structure conditions (7) together with (8) and (9)–(14) imply the ellipticity condition (5) with $\lambda = \Lambda_1/(2^{p+1}p)$ and $F \equiv C_1 \left(\sum_{\nu=1}^{N} |F^\nu|\right)^{p/p-1} + C_2$, where $C_{1,2}$ are numbers depending only on the data.

Concerning right–hand members $B^\mu(x,r,s)$, we assume that these are measurable functions of the form $\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}$, satisfying the conditions:

$$\exists\, \varepsilon \in \left(0, \frac{p^2(1-\kappa_1)}{(n+p)}\right],\ \Lambda_3 > 0 \mid \forall s_i^\mu \in \mathbb{R}^{Nn},\ \forall r_\mu \in \mathbb{R}^N,\ \forall x \in \mathbb{R}^n;\ \mu = 1, ..., N;$$

$$|B^\mu(x,r,s)| \leq \Lambda_3 |s|^\varepsilon. \tag{15}$$

In sequel, we use the notations:

$$\widetilde{u_0^\mu} = \begin{cases} u_0^\mu(x), & x \in \Omega, \quad t = 0, \\ g^\mu(x,t), & x \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

Let $\widetilde{W}(Q) = L^{p'}(W^{1,p'}(0,T);\Omega) \bigcap L^p(0,T;W^{1,p}(\Omega))$, $p' = \dfrac{p}{p-1}$; i. e. the function u belongs to $\widetilde{W}(Q)$ if the integral $\int_0^T \int_\Omega (|u_t|^{p'} + |\nabla u|^p + |u|^p + |u|^{p'})$ is finite.

At last, we suppose that the functions $g^\mu(x,t)$, $u_0^\mu(x)$ in boundary data (3) satisfy the assumptions: $\widetilde{u_0^\mu} \in \widetilde{W}(\partial Q)$; $g^\mu(x,t) \in L^\infty(S)$, $u_0^\mu(x) \in L^\infty(\overline{\Omega} \times \{0\})$.

3. One integral estimate of the sum of squares. For the subsequent considerations we need an integral estimate of the problem (2), (3) solution.

Theorem 1. *Let* $(u^1, \ldots, u^N)$ *be a solution of the problem* (2), (3) *and the assumptions* (7),(8), (9)–(14), (15) *are satisfied. Then there exits a constant* C, *depending only on* f, F, $||\widetilde{u_0^\mu}||_{\widetilde{W}(\partial Q)}$, p, n, ε, η_0, ξ_0, κ_1, Λ_1, Λ_2, $\operatorname{mes} Q$ *and independent of* u^μ *such that*

$$\sum_{\nu=1}^{N} \sup_{0<t<T} \int_\Omega |u^\nu - \widetilde{u_0^\nu}|^2 + \sum_{\nu=1}^{N} \int_0^T \int_\Omega \left(|\nabla(u^\nu - \widetilde{u_0^\nu})|^p\right) \leq C,$$

$$\sum_{\nu=1}^{N} \int_0^T \int_\Omega (|\nabla u^\nu|^p) \leq C,$$

where $\widetilde{u_0^\mu}$ *denotes the values of a function on the parabolic boundary* ∂Q, *belonging to* $\widetilde{W}(Q)$.

P r o o f. Multiplying each equation of (2) by $(u^\mu - \widetilde{u_0^\mu})$ and adding its, after integrating over the domain $\Omega \times (0,t)$, we obtain

$$\sum_{\nu=1}^{N} \int\limits_{\Omega(t)} \frac{1}{2}(u^\nu - \widetilde{u_0^\nu})^2 + \sum_{\nu=1}^{N} \int\limits_0^t \int\limits_\Omega \vec{A}^{(\nu)} \nabla (u^\nu - \widetilde{u_0^\nu}) \le$$

$$\le \sum_{\nu=1}^{N} \int\limits_0^t \int\limits_\Omega |B^{(\nu)}||u^\nu - \widetilde{u_0^\nu}| + \sum_{\nu=1}^{N} \int\limits_0^t \int\limits_\Omega |\widetilde{u_{0t}^\nu}||u^\nu - \widetilde{u_0^\nu}|. \quad (16)$$

Above the initial condition is taken into account. Using the ellipticity condition (5) and the growth conditions of A^μ (6), we have

$$\sum_{\nu=1}^{N} \int\limits_0^t \int\limits_\Omega \left(\vec{A}^{(\nu)} \nabla (u^\nu - \widetilde{u_0^\nu}) \right) = \int\limits_0^t \int\limits_\Omega \left(\sum_{\nu=1}^{N} \vec{A}^{(\nu)} \nabla u^\nu - \sum_{\nu=1}^{N} \vec{A}^{(\nu)} \nabla \widetilde{u_0^\nu} \right) \ge$$

$$\ge \int\limits_0^t \int\limits_\Omega \frac{1}{2}\lambda \left(\sum_{\nu=1}^{N} |\nabla (u^\nu - \widetilde{u_0^\nu})|^p \right) - \int\limits_0^t \int\limits_\Omega \widetilde{C}(p,\lambda) \left(\sum_{\nu=1}^{N} |\nabla u_0^\nu|^p \right) - C.$$

Here we also use the Young's inequality and the following inequality

$$|a+b|^p \le C(p)(|a|^p + |b|^p) \quad \forall a,b \in \mathbb{R}. \quad (17)$$

The first group of terms on the right in (16) by Young's and Sobolev's inequalities, using the condition (15), can be estimated, like as:

$$\sum_{\nu=1}^{N} \int\limits_0^t \int\limits_\Omega |B^{(\nu)}||u^\nu - \widetilde{u_0^\nu}| \le \int\limits_0^t \int\limits_\Omega \left(\sum_{\nu=1}^{N} |\nabla u^\nu| \right)^\varepsilon \left(\sum_{\nu=1}^{N} |u^\nu - \widetilde{u_0^\nu}| \right) \le$$

$$\le \delta_1 \int\limits_0^t \int\limits_\Omega \left(\sum_{\nu=1}^{N} |\nabla (u^\nu - \widetilde{u_0^\nu})| \right)^p + C.$$

Here it has been taken into account that $\varepsilon/p + 1/p < 1/2 + 1/p \le 1$. Taking into account our assumptions and applying Hölder's and Young's inequalities, the last group of integrals in the right–hand side of (16) can be estimated as:

$$\int\limits_0^t \int\limits_\Omega \sum_{\nu=1}^{N} |\widetilde{u_{0t}^\nu}||u^\nu - \widetilde{u_0^\nu}| \le \int\limits_0^t \int\limits_\Omega \left(\sum_{\nu=1}^{N} |\widetilde{u_{0t}^\nu}| \right) \left(\sum_{\nu=1}^{N} |u^\nu - \widetilde{u_0^\nu}| \right) \le$$

$$\le \delta_2 \int\limits_0^t \int\limits_\Omega \left(\sum_{\nu=1}^{N} |\nabla (u^\nu - \widetilde{u_0^\nu})| \right)^p + C\left(\text{mes}Q, \delta_2, \widetilde{u_0^\mu} \right).$$

Collecting the above estimates, from (16) it follows that

$$\int\limits_{\Omega(t)} \frac{1}{2} \left[\sum_{\nu=1}^{N} (u^\nu - \widetilde{u_0^\nu})^2 \right] + \int\limits_0^t \int\limits_\Omega \frac{1}{2}\lambda \left(\sum_{\nu=1}^{N} |\nabla (u^\nu - \widetilde{u_0^\nu})|^p \right) \le$$

$$\le \delta_3 \int\limits_0^t \int\limits_\Omega \left(\sum_{\nu=1}^{N} |\nabla (u^\nu - \widetilde{u_0^\nu})|^p \right) + C(\text{mes}Q, F^\mu, \delta_3, \widetilde{u_0^\mu}).$$

Putting $\delta_3 = \dfrac{1}{4}\lambda$, we get

$$\int\limits_{\Omega(t)} \frac{1}{2}\left[\sum_{\nu=1}^{N}(u^\nu - \widetilde{u_0^\nu})^2\right] + \int\limits_0^t \int\limits_\Omega \frac{1}{4}\lambda\left(\sum_{\nu=1}^{N}|\nabla(u^\nu - \widetilde{u_0^\nu})|^p\right) \le$$

$$\le C\left(\mathrm{mes}Q, F^\mu, \delta_3, \widetilde{u_0^\mu}\right). \quad (18)$$

Taking the supremum by t in the left–hand side of (18), we obtain

$$\sup_{0<t<T}\int\limits_\Omega \sum_{\nu=1}^{N}|u^\nu - \widetilde{u_0^\nu}|^2 + \int\limits_0^T \int\limits_\Omega \left(\sum_{\nu=1}^{N}|\nabla(u^\nu - \widetilde{u_0^\nu})|^p\right) \le C,$$

where the constant C depends from n, p, ε, λ, F, $\mathrm{mes}\,Q$, $||\widetilde{u_0^\mu}||_{\widetilde{W}(\partial Q)}$. Hence, the second statement of Theorem 1 is true.

4. L^∞**–norm estimates.** Let us now turn our attention to the question of boundedness of the solutions to a system whose coefficients satisfy structure hypotheses (7)–(14) and whose right–hand sides satisfy (15). Our main result is the following.

Theorem 2. *Let $(u^1, \dots, u^N)$ be a solution of system (2). If there are the numbers $\alpha^{\mu\nu}$, satisfying the assumptions (7), then for the N linearly independent functions H^1, $\dots$, H^N, defining by (1), there exist constants $C^1, \dots, C^N$ such that the following estimates hold*

$$||H^1||_{L_\infty(Q)} \le C^1, \quad \dots, \quad ||H^N||_{L_\infty(Q)} \le C^N,$$

where the constants $C^1, \dots, C^N$ are depended of the constants in the embedding theorems and of p, n, ε, λ, η_0, ξ_0, σ, ω, Λ_1, Λ_2, f^μ, F^μ; $|g^\mu|_{\infty,(S)}$, $|u_0^\mu|_{\infty,(\Omega)}$, Q, and these are independent of u^μ.

For the components of the solution the same estimates also hold, namely

$$||u^1||_{L_\infty(Q)} \le C^1, \quad \dots \quad ||u^N||_{L_\infty(Q)} \le C^N.$$

To prove of Theorem 2 we need the following well–known Stampacchia's lemma [1, p. 8, lemma 4.1].

Lemma 1. *Let $\psi(y)$ be a nonnegative nondecreasing function defined on $[l_0, \infty)$ which satisfies the condition*

$$\psi(m) \le \frac{C}{(m-l)^\vartheta}\{\psi(l)\}^\delta, \quad \textit{for all} \quad m > l \ge l_0,$$

where $\vartheta > 0$ and $\delta > 1$. Then

$$\psi(l_0 + d) = 0,$$

where $d = C^{1/\vartheta}\{\psi(l_0)\}^{(\delta-1)/\vartheta}2^{\delta/(\delta-1)}$.

We also use the following lemma (see [2, p. 7, prop. 3.1]):

Lemma 2. *If $u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ then there holds the following inequality*

$$\int\limits_0^T\int\limits_\Omega u^q \le C\left(\int\limits_0^T\int\limits_\Omega |\nabla u|^p\right)\left(\operatorname{ess}\sup_{0<t<T}\int\limits_\Omega |u|^2\right)^{p/n},$$

where $q = p\dfrac{n+2}{n}$ and the constant C depends only of p and n.

P r o o f of Theorem 2. Let $\alpha^{11}, \ \ldots, \ \alpha^{N1}$ satisfy the assumptions (7). Let us multiply each μ-th equation of (2) by $\alpha^{\mu 1}$ and add all together. Choosing the testing function, as

$$\operatorname{sign}\left(\sum_{\nu=1}^{N}\alpha^{\nu 1}u^{\nu}\right)\left(\left|\sum_{\nu=1}^{N}\alpha^{\nu 1}u^{\nu}\right|-l\right)_{+} := \operatorname{sign}\left(H^{1}\right)\left(\left|H^{1}\right|-l\right)_{+},$$

where $l \geq l_0 = \max\left[\left\|\sum_{\nu=1}^{N}\alpha^{\nu 1}g^{\nu}\right\|_{L^{\infty}(S)}, \ \left\|\sum_{\nu=1}^{N}\alpha^{\nu 1}u_0^{\nu}\right\|_{L^{\infty}(\Omega)}\right]$, we integrate by t from 0 to t $(t \leq T)$ and by x over the domain Ω. Using the assumptions (7), we have $\int_0^t\int_\Omega\left\langle\sum_{\nu=1}^{N}\alpha^{\nu 1}\vec{A}^{(\nu)}, \nabla H^1\right\rangle =$

$$= \int_0^t\int_\Omega\left\langle\left[\sum_{\nu=1}^{N}\alpha^{\nu 1}\vec{A}^{(\nu)} - \lambda^1\left(\sum_{\nu=1}^{N}\alpha^{\nu 1}\nabla u^{\nu}\right)\right] + \lambda^1\left(\sum_{\nu=1}^{N}\alpha^{\nu 1}\nabla u^{\nu}\right), \nabla H^1\right\rangle \geq$$

$$\geq \int_0^t\int_\Omega \lambda^1\left\langle\sum_{\nu=1}^{N}\alpha^{\nu 1}\nabla u^{\nu}, \nabla H^1\right\rangle - \int_0^t\int_\Omega\left(\xi^1|\nabla H^1| + |F^1||\nabla H^1|\right).$$

From here it follows the inequality

$$\frac{1}{2}\int_{\Omega(t)}(H^1)^2 + \int_0^t\!\!\int_\Omega \lambda_1\left\langle\sum_{\nu=1}^{N}\alpha^{\nu 1}\nabla u^{\nu}, \nabla H^1\right\rangle \leq \int_0^t\!\!\int_\Omega\left(|F^1|+\xi_1\right)|\nabla H^1| + C\int_0^t\!\!\int_\Omega BH^1,$$

where $B := \sum_{\nu=1}^{N}\alpha^{\nu 1}B^{\nu}, \ \ C = C(\alpha^{11}, ..., \alpha^{N1}, p, n)$ is a constant and $\lambda, \ \xi$ are functions from (7). Since $t \in (0,T]$ is arbitrary, taking supremum and using the assuptions, we obtain

$$\sup_{0<t<T}\int_\Omega(H^1)^2 + C_1\int_0^T\int_\Omega|\nabla H^1|^p \leq \int_0^T\int_\Omega\left(|F^1|+\xi^1\right)|\nabla H^1| + C\int_0^T\int_\Omega BH^1, \quad (19)$$

where $C_1 = C_1(\Lambda^1, \alpha^{11}, ..., \alpha^{N1}, p, n)$ and the assumption (8) concerning the function λ_1 has been used. Consequently applying to the terms on the right the generalized Hölder's inequality, we have

$$\int_0^T\int_\Omega|F^1||\nabla H^1| \leq \|\nabla H^1\|_{p,Q}\|F^1\|_{\sigma,Q}\left(\int_0^T\int_\Omega\chi_{A(l)}\right)^{1-1/p-1/\sigma}, \quad (20)$$

$$\int_0^T\int_\Omega\xi^1|\nabla H^1| \leq \|\nabla H^1\|_{p,Q}\|\xi^1\|_{p/\omega,Q}\left(\int_0^T\int_\Omega\chi_{A(l)}\right)^{1-1/p-\omega/p}, \quad (20a)$$

$$\int_0^T\int_\Omega BH^1 \leq \|H^1\|_{q,Q}\|B\|_{p/\varepsilon,Q}\left(\int_0^T\int_\Omega\chi_{A(l)}\right)^{1-1/q-\varepsilon/p}, \quad (20b)$$

where $\chi_{A(l)}$ is the characteristic function of the set $A(l)$. From the conditions (10), (9), (15) and Theorem 1 imply that

$$\|F^1\|_{\sigma,Q} \leq C_2, \qquad \|\xi^1\|_{p/\omega,Q} \leq C_3, \qquad \|B\|_{p/\varepsilon,Q} \leq C_4. \tag{21}$$

Collecting (20)-(20b) and taking into account the (21), from (19) we obtain the inequality

$$\sup_{0<t<T} \int_\Omega (H^1)^2 + \int_0^T \int_\Omega |\nabla H^1|^p \leq C_1 \|\nabla H^1\|_{p,Q} \{\psi(l)\}^{1-1/p-1/\sigma} + \\ + C_2 \|\nabla H^1\|_{p,Q} \{\psi(l)\}^{1-1/p-\omega/p} + C_3 \|H^1\|_{q,Q} \{\psi(l)\}^{1-1/q-\varepsilon/p}, \tag{22}$$

where it is denote $\psi(l) := \int_0^T \mathrm{mes} A\{H^1 \geq l\}(l,t)dt$. From Lemma 2 it follows

$$||H^1||_{q,Q} \leq \left(\sup_{0<t<T} \int_\Omega (H^1)^2 + \int_0^T \int_\Omega |\nabla H^1|^p \right)^{\frac{p+n}{qn}}. \tag{23}$$

From the relation (22) and the previous inequality, we get

$$\sup_{0<t<T} \int_\Omega (H^1)^2 + \int_0^T \int_\Omega |\nabla H^1|^p \leq \\ \leq C_1 \left(\sup_{0<t<T} \int_\Omega (H^1)^2 + \int_0^T \int_\Omega |\nabla H^1|^p \right)^{1/p} \{\psi(l)\}^{1-1/p-1/\theta} + \\ + C_2 \left(\sup_{0<t<T} \int_\Omega (H^1)^2 + \int_0^T \int_\Omega |\nabla H^1|^p \right)^{1/p} \{\psi(l)\}^{1-1/p-\omega/p} + \\ + C_3 \left(\sup_{0<t<T} \int_\Omega (H^1)^2 + \int_0^T \int_\Omega |\nabla H^1|^p \right)^{(n+p)/nq} \{\psi(l)\}^{1-1/q-\varepsilon/p}. \tag{24}$$

Applying Young's inequality to the right–hand side of (24), we get

$$\sup_{0<t<T} \int_\Omega (H^1)^2 + \int_0^T \int_\Omega |\nabla H^1|^p \leq C_1 \{\psi(l)\}^{(1-1/p-1/\sigma)\left(\frac{p}{p-1}\right)} + \\ + C_2 \{\psi(l)\}^{(1-1/p-\omega/p)\left(\frac{p}{p-1}\right)} + C_3 \{\psi(l)\}^{(1-1/q-\varepsilon/p)\left(\frac{nq}{n+p}\right)^{\#}};$$

where $C_{1,2,3} := C\left(\widetilde{u_0}, \widetilde{v_0}, \widetilde{w_0}, F, \sigma, \omega, \Lambda_1, \Lambda_2, p, n\right)$ and $\left(\dfrac{nq}{n+p}\right)^{\#}$ is such that $\left(\left(\dfrac{nq}{n+p}\right)^{\#}\right)^{-1} + \dfrac{n+p}{nq} = 1$. Applying again (23), we have

$$\left(\|H^1\|_{q,Q}\right)^{nq/(n+p)} \leq C_1 \{\psi(l)\}^{(1-1/p-1/\sigma)\left(\frac{p}{p-1}\right)} + \\ + C_2 \{\psi(l)\}^{(1-1/p-\omega/p)\left(\frac{p}{p-1}\right)} + C_3 \{\psi(l)\}^{(1-1/q-\varepsilon/p)\left(\frac{nq}{n+p}\right)^{\#}}. \tag{25}$$

Let us use the estimation

$$(m-l)\{\psi(m)\}^{1/q} = (m-l)\left(\int_0^T\int_\Omega \chi_{A(m)}\right)^{1/q} < \left(\int_0^T\int_\Omega (H^1)^q \chi_{A(m)}\right)^{1/q} < ||H^1||_{q,Q},$$

where $m > l \geq l_0$. Substituting this into (25), we come to the inequality

$$\psi(m) \leq \frac{C_1}{(m-l)^q}\{\psi(l)\}^{\delta_1} + \frac{C_2}{(m-l)^q}\{\psi(l)\}^{\delta_2} + \frac{C_3}{(m-l)^q}\{\psi(l)\}^{\delta_3} \tag{26}$$

where $\delta_1 = \left(1 - \frac{1}{p} - \frac{1}{\sigma}\right)\left(\frac{p(n+p)}{n(p-1)}\right)$, $\delta_2 = \left(1 - \frac{1}{p} - \frac{\omega}{p}\right)\left(\frac{p(n+p)}{n(p-1)}\right)$, and $\delta_3 = \left(1 - \frac{n}{p(n+2)} - \frac{\varepsilon}{p}\right)\Big/\left(\frac{n}{n+p} - \frac{n}{p(n+2)}\right)$. From the assumptions (10) on F^μ, it follows that

$$1 - \frac{1}{p} - \frac{1}{\sigma} > \frac{n(p-1)}{p(n+p)}, \qquad \text{thus} \quad \delta_1 > 1.$$

From the assumptions (9) on ξ_j, it follows that

$$1 - \frac{1}{p} - \frac{\omega}{p} > \frac{n(p-1)}{p(n+p)}, \qquad \text{thus} \quad \delta_2 > 1.$$

From the assumptions (15) on B^μ, it follows that

$$1 - \frac{n}{p(n+2)} - \frac{\varepsilon}{p} > \frac{n}{n+p} - \frac{n}{p(n+2)}, \qquad \text{thus} \quad \delta_3 > 1.$$

Without loss of generality, we can assume that $\psi(l) < 1$. In fact, from the first statement of Theorem 1 and (23) it follows that

$$(l-l_0)\{\psi(l)\}^{1/q} = (l-l_0)\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1/q} < \left(\int_0^T\int_\Omega (H^1-l_0)^q \chi_{A(l)}\right)^{1/q} <$$

$$< ||H^1 - l_0||_{q,Q} \leq \left(\sup_{0<t<T}\int_\Omega (H^1-l_0)^2 + \int_0^T\int_\Omega |\nabla(H^1-l_0)|^p\right)^{\frac{p+n}{qn}} \leq \widetilde{C},$$

where $l \geq l_0$. Hence $\psi(l) \leq \widetilde{C}^q/(l-l_0)^q$, and it is easy to see that $\psi(l) < 1$, whenever $l > \widetilde{C} + l_0$. Since $\psi(l)$ is non-increasing function, $\psi(l) < 1$ is true for all $l > \widetilde{C} + l_0$. Due to this, from (26) it follows

$$\psi(m) \leq \frac{C}{(m-l)^q}\{\psi(l)\}^\delta, \tag{27}$$

where $\delta := \min[\delta_1, \delta_2, \delta_3]$ and $C := \max[C_1, C_2, C_3]$. Using Lemma 1, the relation (27) implies $\psi(l_0 + d) = 0$ for some sufficiently large finite number d, depending only of the constants in the embedding theorems and of p, n, ε, λ, η_0, ξ_0, σ, ω, Λ_1, Λ_2, F^μ, $|g^\mu|_{\infty,(S)}$, $|u_0^\mu|_{\infty,(\Omega)}$, Q and independent of u^μ. Analogously, it is done to the rest linear combinations $H^2 = \sum_{\nu=1}^N \alpha^{\nu 2} u^\nu$, $\ldots$, $H^N = \sum_{\nu=1}^N \alpha^{\nu N} u^\nu$.

As it is easy to see, from the previous reasonings immediately follows that the same estimates hold for the components $u^1, \ldots, u^N$ of the solution. In fact,

$$||u^\mu||_\infty = ||\mathrm{Det}(C^\mu)||_\infty \, \left|(\mathrm{Det}(A))^{-1}\right| \le C\left(\alpha^{11}, \ldots, \alpha^{NN}, ||H^1||_\infty, \ldots, ||H^N||_\infty\right),$$

where C^μ denotes the matrix, obtained from A, by replacing of the μ–th column and $H^1, \ldots, H^N$, i.e.

$$C^\mu = \begin{pmatrix} \alpha^{11} & \cdots & \alpha^{(\mu-1)1} & H^1 & \alpha^{(\mu+1)1} & \cdots & \alpha^{1N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha^{1N} & \cdots & \alpha^{(\mu-1)N} & H^N & \alpha^{(\mu+1)N} & \cdots & \alpha^{NN} \end{pmatrix}.$$

Hence, from here the statement follows.

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ОБМЕЖЕНІСТЬ СЛАБКИХ РОЗВ'ЯЗКІВ НЕДІАГОНАЛЬНОЇ СИНГУЛЯРНОЇ ПАРАБОЛІЧНОЇ СИСТЕМИ РІВНЯНЬ

Вивчається обмеженість слабких розв'язків для недіагональної параболічної системи сингулярних квазілінійних диференціальних рівнянь з матрицею коефіцієнтів, що задовольняє спеціальні структурні умови. Для цього застосовується техніка, що базується на оцінці лінійних комбінацій невідомих.

ОГРАНИЧЕННОСТЬ СЛАБЫХ РЕШЕНИЙ НЕДИАГОНАЛЬНОЙ СИНГУЛЯРНОЙ ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ УРАВНЕНИЙ

Изучается ограниченность слабых решений недиагональной параболической системы сингулярных квазилинейных дифференциальных уравнений с матрицей коэффициентов, удовлетворяющей специальным структурным условиям. Для этого применяется метод, основывающийся на оценке линейных комбинаций неизвестных.

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