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O. Lyzun

ENTIRE FUNCTIONS WITH PRESCRIBED GROWTH

An entire function f with prescribed growth of its main characteristics is constructed in a form of certain infinite product such that

$$N(r,f) \sim T(r,f) \sim \log m_q(r,f) \sim \log M(r,f) \sim m_2(r, \log|f|) \sim \lambda(r), \quad r \to +\infty,$$

where m_q are q-integral means and $\lambda(r)$ is a positive, continuous, increasing to $+\infty$ and convex with respect to $\log r$ function.

Introduction. In 1966 J.Clunie and J.Kovari [1] constructed an entire function f with prescribed growth of its logarithm of the maximum modulus $\log M(r,\, f)$, the Nevanlinna characteristic $T(r,\, f)$ and Nevanlinna counting function of c-points for any $c \in \mathbb{C}$.

The main result of their paper is the following:

Theorem B. *Let $\lambda(r)$ be an arbitrary real function, increasing and convex with respect to $\log r$ and such that $\lambda(r) \neq O(\log r)$, $r \to +\infty$. Then there exists an entire function $f(z)$ satisfying*

1) $\log M(r,f) \sim \lambda(r), \quad r \to \infty,$

2) $T(r,f) \sim \lambda(r), \quad r \to \infty,$

and also

3) $N(r,c,f) \sim \lambda(r), \quad r \to +\infty$

for any constant $c \in \mathbb{C}$,

where $N(r,c,f)$ is the Nevanlinna counting function of c-points.

An entire function f with the given above properties is represented by a certain power series in [1].

In the theory of entire and meromorphic functions the square means of their modulus logarithms $m_2(r,\, \log|f|)$ plays an important role. In particular, for the meromorphic function of finite order ρ J. Miles and D. Shea [4] have established the following best possible estimate

$$\limsup_{r\to\infty} \frac{N(r,f,\infty)+N(r,f,0)}{m_2(r,\log|f|)} \geq \frac{|\sin\pi\rho|}{\pi\rho} \left\{ \frac{2}{1+\frac{\sin 2\pi\rho}{2\pi\rho}} \right\}^{1/2}, \tag{1}$$

where

$$\rho := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

is the order of f and

$$m_2(r,\log|f|) := \left(\frac{1}{2\pi} \int_0^{2\pi} |\log|f(re^{i\theta})||^2 d\theta \right)^{1/2}.$$

Using (1), in [4] the best estimate known so far for the value

$$\varkappa(f) = \limsup_{r\to\infty} \frac{N(r,f,\infty)+N(r,f,0)}{T(r,f)},$$

has been determined, namely,

$$\varkappa(f) \geq 0.9 \frac{|\sin\pi\rho|}{\rho+1}, \qquad 1 < \rho < +\infty,$$

where f is a meromorphic function of order ρ.

The best possible lower estimate for $\varkappa(f)$ is unknown. The problem was posed by R. Nevanlinna.

We use the following notations. A positive function $\lambda(r)$, continuous on $[0,+\infty)$, with $\lambda(0)=0$ and strictly increasing to $+\infty$ is called a growth function. We assume here that $\lambda(r)$ is convex with respect to $\log r$. So,

$$\lambda(r)=\int_0^r \frac{\mu(t)}{t}dt, \qquad r\to+\infty, \tag{2}$$

where $\mu(r)$ is continuous, strictly increasing with $\lim_{r\to\infty}\mu(r)=\infty$.

In this paper we construct an entire function f with prescribed growth represented by a certain infinite product such that

$$N(r,f)\sim T(r,f)\sim \log m_q(r,f)\sim \log M(r,f)\sim m_2(r,\log|f|)\sim\lambda(r), \quad r\to+\infty,$$

where

$$m_q(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^q d\theta\right)^{1/q}, \qquad q\geq 1.$$

For an entire function f, $f(0)=1$, we denote the sequence of its zeroes by $\{a_j\}$, $\alpha_j=\arg a_j$.

Let

$$n_k(r)=n_k(r,f):=\sum_{|a_j|\leq r} e^{-ik\alpha_j}, \qquad k\in\mathbb{Z}, \quad r>0, \tag{3}$$

$$n(r)=n(r,f)=n_0(r,f), \tag{4}$$

$$N(r,f)=\int_0^r \frac{n(t,f)}{t}dt \tag{5}$$

and

$$\log f(z)=\sum_{k\in\mathbb{N}}\gamma_k z^k$$

be the development of $\log f$, $\log f(0)=0$, in some neighbourhood of the origin and

$$c_k(r,f)=\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\log|f(re^{i\theta})|d\theta, \quad k\in\mathbb{Z}.$$

The main result of this paper is the following. Let λ be a growth function and μ determined by the relation (2).

Theorem. *If for arbitrary $\varepsilon>0$ there exists a number r_0 such that*

$$\mu(r(1+\varepsilon))<\varepsilon\left(\frac{\sqrt{\mu(r)}\lambda(r)}{\log\mu(r)}\right), \tag{6}$$

for all $r>r_0$, then there exists an entire function f_μ such that

$$N(r,f_\mu)\sim T(r,f_\mu)\sim\log m_q(r,f_\mu)\sim\log M(r,f_\mu)\sim m_2(r,\log|f_\mu|)\sim\lambda(r),$$

$r\to+\infty$.

P r o o f. We construct a function f_μ by the form

$$f_\mu(z) = \prod_{j\in\mathbb{N}} \left(1 - \left(\frac{z}{r_j}\right)^{2j-1}\right), \tag{7}$$

where

$$r_j = \mu^{-1}(j^2). \tag{8}$$

Since the function $\mu(r)$ increases to $+\infty$, then $r_j \to +\infty$ too. Therefore the series $\sum_{j=1}^{\infty} \left(\frac{z}{r_j}\right)^{2j-1}$ converges as well as the product (7) uniformly on compact subsets of $\mathbb{C}$. So, $f = f_\mu$ is an entire function.

We show that

$$N(r, f_\mu) \sim \lambda(r), \quad r \to \infty. \tag{9}$$

In fact,

$$n(r_p, f_\mu) = \sum_{j=1}^{p}(2j-1) = p^2, \qquad p \in \mathbb{N}. \tag{10}$$

On the other hand

$$\mu(r_p) = \mu(\mu^{-1}(p^2)) = p^2, \qquad p \in \mathbb{N}. \tag{11}$$

Thus

$$n(r_p, f_\mu) = \mu(r_p), \qquad p \in \mathbb{N}. \tag{12}$$

Let now $r_p \le r < r_{p+1}$. Then

$$n(r_p, f_\mu) = n(r, f_\mu) \le \mu(r) \le \mu(r_{p+1}), \quad r \to +\infty.$$

Hence, using (10), (11), (12) we obtain

$$0 \le \mu(r) - n(r, f_\mu) \le \mu(r_{p+1}) - \mu(r_p) = 2p + 1 = o(\mu(r_p)) = o(\mu(r)), \qquad r \to +\infty.$$

Thus,

$$\mu(r) + o(\mu(r)) \le n(r, f_\mu) \le \mu(r), \qquad r \to +\infty,$$

that is

$$n(r, f_\mu) = \mu(r) + o(\mu(r)), \qquad r \to +\infty. \tag{13}$$

Taking into account (2), (5) and (13), we have

$$N(r, f_\mu) = \int_0^r \frac{\mu(r) + o(\mu(r))}{t} dt = \lambda(r) + o(\lambda(r)), \qquad r \to +\infty. \tag{14}$$

The relation (9) is proved.

Let us show now that

$$m_2(r, \log|f_\mu|) \sim \lambda(r), \qquad r \to +\infty.$$

We have

$$\log|f_\mu(z)| = \log \prod_{j=1}^{\infty} \left|1 - \left(\frac{z}{r_j}\right)^{2j-1}\right| = \sum_{j=1}^{\infty} \log\left|1 - \left(\frac{z}{r_j}\right)^{2j-1}\right| = \tag{15}$$

$$\sum_{j=1}^{\infty} \sum_{m=0}^{2(j-1)} \log\left|1 - \frac{z}{r_j \omega_j^m}\right|,$$

where $\omega_j = \exp\left\{\dfrac{2\pi i}{2j-1}\right\}$. Since

$$\log\left(1 - \frac{z}{r_j\omega_j^m}\right) = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z\bar{\omega_j}^m}{r_j}\right)^k, \qquad |z| < r_1, \tag{16}$$

then inserting (16) into (15) we obtain

$$\log f_\mu(z) = \sum_{k=1}^{\infty}\gamma_k z^k = -\sum_{j=1}^{\infty}\sum_{m=0}^{2(j-1)}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z\bar{\omega_j}^m}{r_j}\right)^k, \qquad |z| < r_1.$$

Collecting the coefficients at z^k we obtain

$$\gamma_k = -\frac{1}{k}\sum_{j=1}^{\infty}\sum_{m=0}^{2(j-1)}\left(\frac{\bar{\omega_j}^m}{r_j}\right)^k. \tag{17}$$

According to [6] (see also [3, p. 10])

$$c_k(r,f) = \frac{1}{2}\gamma_k r^k + \frac{1}{2k}\sum_{|a_j|\le r}\left(\left(\frac{r}{a_j}\right)^k - \left(\frac{\bar{a}_j}{r}\right)^k\right),$$

where a_j are zeroes of f.

Using this equality, (17) and the fact that the zeroes of the function f are of the form $a_j^{(m)} = r_j\omega_j^m$, we obtain

$$c_k(r,f_\mu) = -\frac{r^k}{2k}\sum_{j=1}^{\infty}\sum_{m=0}^{2(j-1)}\left(\frac{\bar{\omega_j}^m}{r_j}\right)^k + \frac{r^k}{2k}\sum_{r_j\le r}\sum_{m=0}^{2(j-1)}\left(\frac{\bar{\omega_j}^m}{r_j}\right)^k -$$

$$\frac{1}{2k}\sum_{r_j\le r}\sum_{m=0}^{2(j-1)}\left(\frac{r_j\bar{\omega_j}^m}{r}\right)^k = -\frac{1}{2k}\sum_{r_j>r}\left(\frac{r}{r_j}\right)^k\sum_{m=0}^{2(j-1)}\bar{\omega_j}^{mk} - \tag{18}$$

$$\frac{1}{2k}\sum_{r_j\le r}\left(\frac{r_j}{r}\right)^k\sum_{m=0}^{2(j-1)}\bar{\omega_j}^{mk}.$$

Taking into consideration that $\log|f|$ is a real function we have also

$$c_{-k}(r,f) = \overline{c_k(r,f)}, \qquad k\in\mathbb{N},$$

$$c_0(r,f) = N(r,f). \tag{19}$$

To estimate $|c_k(r,f)|$ we consider the following sum $\displaystyle\sum_{m=0}^{2(j-1)}\bar{\omega_j}^{mk} = \sum_{m=0}^{2(j-1)}\left(\bar{\omega_j}^k\right)^m$. This is the sum of the geometrical progression with quotient $q = \bar{\omega_j}^k$. Taking into account that $\omega_j = e^{\frac{2i\pi}{2j-1}}$, we consider two cases.

1) $q = 1$, that is $e^{-\frac{2i\pi}{2j-1}k} = 1$. This is true if and only if $\dfrac{k}{2j-1}\in\mathbb{N}$, because of $e^z = 1$ if and only if $z = 2i\pi l$, where $l\in\mathbb{Z}$. We consider l positive. In this case

$$\sum_{m=0}^{2(j-1)}\left(\bar{\omega_j}^k\right)^m = \sum_{m=0}^{2(j-1)}1 = 2j-1.$$

2) $q \neq 1$, that is $e^{-\frac{2i\pi}{2j-1}k} \neq 1$. This is true if and only if $\dfrac{k}{2j-1} \notin \mathbb{N}$. In this case the sum of this progression is

$$\frac{(\bar{\omega_j}^k)^{2j-1} - 1}{\bar{\omega_j}^k - 1} = \frac{(\bar{\omega_j}^{2j-1})^k - 1}{\bar{\omega_j}^k - 1} = 0, \qquad k, j \in \mathbb{N}.$$

Thus,

$$\sum_{m=0}^{2(j-1)} \bar{\omega_j}^{mk} = \begin{cases} 2j-1, & \dfrac{k}{2j-1} \in \mathbb{N}, \\ 0, & \dfrac{k}{2j-1} \notin \mathbb{N}. \end{cases} \tag{20}$$

Using (20) and (3) we obtain

$$n_k(t) = \sum_{\substack{r_j \le t \\ (2j-1)|k}} (2j-1), \tag{21}$$

where $(2j-1)|k$ means that $(2j-1)$ is a divisor of k.

Representing the sums in (18) by the Stieltjes integral, we have

$$|c_k(r,f)| = \frac{1}{2k}\int_0^r \left(\frac{t}{r}\right)^k dn_k(t) + \frac{1}{2k}\int_r^{+\infty} \left(\frac{r}{t}\right)^k dn_k(t) =$$

$$\frac{1}{2}\int_r^{+\infty} \left(\frac{r}{t}\right)^k n_k(t)\frac{dt}{t} - \frac{1}{2}\int_0^r \left(\frac{t}{r}\right)^k n_k(t)\frac{dt}{t} \le \frac{1}{2}\int_r^{+\infty} \left(\frac{r}{t}\right)^k n_k(t)\frac{dt}{t}. \tag{22}$$

We shall now prove that for all $r > 0$, $k \ge 2$ the following estimate is true

$$0 \le n_k(r) \le Ck\log k, \tag{23}$$

where C is a certain positive constant. It follows from (21) that

$$0 \le n_k(r) \le \sum_{(2j-1)|k} (2j-1) \le \sigma(k),$$

where $\sigma(k)$ is a sum of all divisors of the number k.

Using (21), note, that for $k = 1$, $n_k(r) \le 1$. Thus, from (22) $|c_1(r,f)| \le 1$.

An estimate of $\sigma(k)$ may be obtained as follows. While

$$n = p_1^{k_1} p_2^{k_2} \cdots p_\nu^{k_\nu},$$

where $p_1, p_2, ..., p_\nu$ are different primes. We use the multiplicative property of $\sigma(n)$ (see [5]) that is $\sigma(nm) = \sigma(m)\sigma(n)$. So, it is enough to consider the case when $n = p^k$, where p is a prime. Then the divisors are $1, p, p^2, ..., p^k$, hence

$$\sigma(n) = 1 + p + p^2 + ... + p^k = \frac{p^{k+1} - 1}{p - 1},$$

therefore

$$\sigma(n) = \frac{(p_1^{k_1+1} - 1)(p_2^{k_2+1} - 1)\cdots(p_\nu^{k_\nu+1} - 1)}{(p_1 - 1)(p_2 - 1)\cdots(p_\nu - 1)}. \tag{24}$$

In view of (24) we obtain the estimate for sufficiently large n

$$\sigma(n) \le \frac{p_1^{k_1+1} p_2^{k_2+1} \cdots p_\nu^{k_\nu+1}}{(p_1 - 1)(p_2 - 1)\cdots(p_\nu - 1)} = n\left(\frac{p_1}{p_1 - 1}\cdots\frac{p_\nu}{p_\nu - 1}\right) =$$

$$n\left(\frac{1}{1-\frac{1}{p_1}}\cdots\frac{1}{1-\frac{1}{p_\nu}}\right)=\frac{n}{\prod\limits_{j=1}^{n}\left(1-\frac{1}{p_j}\right)}\le\frac{n}{\prod\limits_{p\le n}\left(1-\frac{1}{p}\right)}=$$

$$n\log n\left(\frac{1}{C_0\left(1+O\left(\frac{1}{\log n}\right)\right)}\right)\le Cn\log n,\qquad C<1,$$

Here we used a known result [7]

$$\prod_{p\le n}\left(1-\frac{1}{p}\right)=\frac{C_0}{\log n}\left(1+O\left(\frac{1}{\log n}\right)\right),$$

where C_0 is a constant, $C_0>1$. So (23) is proved.

Let $r_{m-1}\le r<r_m$, $m\ge 2$. We have

$$\sum_{k=1}^{\infty}|c_k(r,f)|^2=\sum_{1\le k\le m^2}|c_k(r,f)|^2+\sum_{k>m^2}|c_k(r,f)|^2. \tag{25}$$

Taking into account (22) and (23) we obtain the following estimate

$$|c_k(r,f_\mu)|\le r^k\int\limits_r^\infty\frac{n_k(t)}{t^{k+1}}dt\le Cr^kk\log k\int\limits_r^\infty\frac{dt}{t^{k+1}}=C\log k,\quad k\ge 2. \tag{26}$$

Thus, from (26) and (8)

$$\sum_{k\le m^2}|c_k(r,f)|^2\le 1+C^2\sum_{2\le k\le m^2}(\log k)^2\le C^2m^2(\log m)^2\le\frac{C^2}{4}(\sqrt{\mu(r_m)})^2\log^2\mu(r_m),$$

where by C we denote some constants (possibly different). But

$$\frac{\mu(r_m)}{\mu(r_{m-1})}=\frac{m^2}{(m-1)^2}\le 4,\quad m\ge 2,$$

then

$$\frac{C^2}{4}(\sqrt{\mu(r_m)})^2\log^2\mu(r_m)\le C_1^2\mu(r_{m-1})\log^2\mu(r_{m-1})\le C_1^2\mu(r)\log^2\mu(r),$$

here C_1 is a positive constant. Thus,

$$\sum_{k\le m^2}|c_k(r,f)|^2\le C_1^2\mu(r)\log^2\mu(r). \tag{27}$$

Now consider the second sum in (25). In view of (22), we obtain

$$|c_k(r,f_\mu)|\le r^k\int\limits_r^\infty\frac{n_k(t)}{t^{k+1}}dt=r^k\left(\int\limits_r^{rk^{1/k}}\frac{n_k(t)}{t^{k+1}}dt+\int\limits_{rk^{1/k}}^\infty\frac{n_k(t)}{t^{k+1}}dt\right)=I_1+I_2.$$

Let us estimate I_1. As $k>m^2$ then using (23) we have

$$r^k\int\limits_r^{rk^{1/k}}\frac{n_k(t)}{t^{k+1}}dt\le r^kn_k(rk^{1/k})\int\limits_r^{rk^{1/k}}\frac{dt}{t^{k+1}}\le\frac{n(rk^{1/k})}{k}\le\frac{\log kn(rm^{2/m^2})}{k^{1/2}k^{1/2}\log k}\le$$

$$\frac{1}{\sqrt{k}\log k}n(rm^{2/m^2})C\frac{\log m}{m} \le C_1\frac{1}{\sqrt{k}\log k}\frac{\log\mu(r_m)}{\sqrt{\mu(r_m)}}\mu\left(r\mu(r_m)^{1/\mu(r_m)}\right) \le$$

$$C_1\frac{1}{\sqrt{k}\log k}\frac{\log\mu(r)}{\sqrt{\mu(r)}}\mu\left(r\mu(r)^{1/\mu(r)}\right),$$

where C, C_1 are some positive constants.

Now we estimate I_2. Taking into account (23), we obtain

$$r^k\int\limits_{rk^{1/k}}^{\infty}\frac{n_k(t)}{t^{k+1}}dt \le \frac{C_2 k\log k}{k^2} = \frac{C_2\log k}{k}.$$

Thus,

$$\sum_{k>m^2}|c_k(r,f)|^2 \le \sum_{k>m^2}\left(C_1\frac{1}{\sqrt{k}\log k}\frac{\log\mu(r)}{\sqrt{\mu(r)}}\mu\left(r\mu(r)^{1/\mu(r)}\right) + C_2\frac{\log k}{k}\right)^2. \quad (28)$$

From (25), (27) and (28) we obtain

$$\sum_{k=1}^{\infty}|c_k(r,f)|^2 = |c_1^2(r,f)| + \sum_{k=2}^{\infty}|c_k(r,f)|^2 \le 1 + C\mu(r)\log^2\mu(r) + \sum_{k>m^2}|c_k(r,f)|^2 \le$$

$$C_0\mu(r)\log^2\mu(r) + \sum_{k>m^2}\left(C_1\frac{1}{\sqrt{k}\log k}\frac{\log\mu(r)}{\sqrt{\mu(r)}}\mu\left(r\mu(r)^{1/\mu(r)}\right) + C_2\frac{\log k}{k}\right)^2 =$$

$$C_0\mu(r)\log^2\mu(r) + C_1^2\mu^2\left(r\mu(r)^{1/\mu(r)}\right)\frac{\log^2\mu(r)}{\mu(r)}\sum_{k>m^2}\frac{1}{k\log^2 k} +$$

$$2C_1C_2\mu\left(r\mu(r)^{1/\mu(r)}\right)\frac{\log\mu(r)}{\sqrt{\mu(r)}}\sum_{k>m^2}\frac{1}{k^{3/2}} + C_2^2\sum_{k>m^2}\frac{\log^2 k}{k^2} \le \quad (29)$$

$$C_0\mu(r)\log\mu(r) + C_3\mu^2\left(r\mu(r)^{1/\mu(r)}\right)\frac{\log^2\mu(r)}{\mu(r)} + C_4\mu\left(r\mu(r)^{1/\mu(r)}\right)\frac{\log\mu(r)}{\sqrt{\mu(r)}} + C_5 \le$$

$$C_6\mu^2\left(r\mu(r)^{1/\mu(r)}\right)\frac{\log^2\mu(r)}{\mu(r)} \le C_7\mu^2(r(1+\varepsilon))\frac{\log^2\mu(r)}{\mu(r)} = o(\lambda^2(r)) \quad , \quad r\to+\infty.$$

Since the Parseval equality and (19) imply

$$m_2^2(r,\log|f|) = \sum_{k=-\infty}^{\infty}|c_k(r,f)|^2 = N^2(r,f) + 2\sum_{k=1}^{\infty}|c_k(r,f)|^2, \quad (30)$$

then, in view of (14),(25) and (29) we obtain the needed equality

$$m_2(r,\log|f|) = \lambda(r) + o(\lambda(r)) \quad , \quad r\to+\infty.$$

The relation $T(r,f)\sim\lambda(r),\quad r\to+\infty$ follows from (14),(30) and the inequalities

$$N(r,f) \le T(r,f) \le m_1(r,\log|f|) \le m_2(r,\log|f|).$$

We now show that

$$\log M(r,f)\sim\lambda(r), \quad r\to+\infty. \quad (31)$$

We have from (7), putting $z = re^{i\theta}$,

$$\log M(r,f) \le \sum_{j=1}^{\infty} \log\left(1 + \left(\frac{r}{r_j}\right)^{2j-1}\right) =$$

$$\sum_{r_j \le r} \log\left(1 + \left(\frac{r}{r_j}\right)^{2j-1}\right) + \sum_{r_j > r} \log\left(1 + \left(\frac{r}{r_j}\right)^{2j-1}\right) = \sum^{(1)} + \sum^{(2)}.$$

Let $r_j \le r$ then

$$\sum^{(1)} = \sum_{r_j \le r} (2j-1) \log \frac{r}{r_j} + \sum_{r_j \le r} \log\left(1 + \left(\frac{r_j}{r}\right)^{2j-1}\right).$$

Integrating by parts, we obtain

$$\sum_{r_j \le r} (2j-1) \log \frac{r}{r_j} = \int_0^r \log \frac{r}{t} dn(t) = \int_0^r \frac{n(t)}{t} dt = N(r).$$

Taking into account (14), we have

$$\sum^{(1)} = (1 + o(1))\lambda(r) + \sum_{r_j \le r} \log\left(1 + \left(\frac{r_j}{r}\right)^{2j-1}\right), \qquad r \to +\infty.$$

Thus

$$\sum_{r_j \le r} \log\left(1 + \left(\frac{r_j}{r}\right)^{2j-1}\right) \le \tilde{n}(r) \log 2 \le \tilde{n}(r),$$

where $\tilde{n}(r)$ is the number of values r_j in $[0, r]$.

Let now $r_j > r$, $a = a(r) = \mu(r)^{\frac{1}{2\sqrt{\mu(r)}}}$, $\qquad r_{m-1} \le r < r_m$. We have

$$\sum^{(2)} = \sum_{r < r_j \le ar} \log\left(1 + \left(\frac{r}{r_j}\right)^{2j-1}\right) + \sum_{ar < r_j} \log\left(1 + \left(\frac{r}{r_j}\right)^{2j-1}\right) \le$$

$$\log 2 \sum_{r < r_j \le ar} 1 + \sum_{ar < r_j} \left(\frac{r}{r_j}\right)^{2j-1}$$

Since

$$\sum_{ar < r_j} \left(\frac{r}{r_j}\right)^{2j-1} \le \sum_{ar < r_j} \left(\frac{1}{a}\right)^{2j-1} \le \sum_{r_{m-1} < r_j} \left(\frac{1}{a}\right)^{2j-1} \le \sum_{m-1 < j} \left(\frac{1}{a}\right)^{2j-1},$$

we obtain

$$\sum^{(2)} \le \tilde{n}(ar) - \tilde{n}(r) + \sum_{m-1 < j} \left(\frac{1}{a}\right)^{2j-1} \le \tilde{n}(ar) - \tilde{n}(r) + \left(\frac{1}{a}\right)^{2(m-1)} a \frac{a^2}{a^2 - 1} \le$$

$$\tilde{n}(ar) - \tilde{n}(r)) + \left(\frac{1}{a}\right)^{2(m-1)} \frac{a^2}{a-1}.$$

Remark that

$$\tilde{n}(ar) = \sum_{r_j \le ar} 1 = \sum_{\mu(r_j) \le \mu(ar)} 1 = \sum_{j^2 \le \mu(ar)} 1 = \sum_{j=1}^{\sqrt{\mu(ar)}} 1 \le \sqrt{\mu(ar)}.$$

Taking into account the relation (6), we find

$$\tilde{n}(ar) \le \frac{\mu(ar)}{\sqrt{\mu(r)}} \le \frac{\mu((1+\varepsilon)r)}{\sqrt{\mu(r)}} \le \varepsilon \frac{\lambda(r)}{\log \mu(r)}, \quad r > r_0. \tag{32}$$

Next

$$\left(\frac{1}{a}\right)^{2(m-1)} \frac{a^2}{a-1} = \mu(r)^{-\frac{2(m-1)}{2\sqrt{\mu(r)}}} \frac{a^2}{a-1} \le$$

$$\mu(r)^{-\sqrt{\frac{\mu(r_m)}{\mu(r)}} + \frac{1}{\sqrt{\mu(r)}}} \frac{4}{a-1} \le \frac{C_8}{(a-1)\mu(r)^{\sqrt{\frac{\mu(r_m)}{\mu(r)}}}} \le$$

$$\frac{C_9\sqrt{\mu(r)}}{\mu(r)^{\sqrt{\frac{\mu(r_m)}{\mu(r)}}} \log \mu(r)} \le \frac{C_9\sqrt{\mu(r)}}{\mu(r)\log \mu(r)} = \frac{C_9}{\sqrt{\mu(r)}\log \mu(r)}.$$

Here C_8, C_9 are some positive constants. Thus,

$$\log M(r,f) \le (1+o(1))\lambda(r) + \tilde{n}(r) + \tilde{n}(ar) - \tilde{n}(r) + \frac{C_9}{\sqrt{\mu(r)}\log \mu(r)} =$$

$$(1+o(1))\lambda(r) + \tilde{n}(ar) + \frac{C_9}{\sqrt{\mu(r)}\log \mu(r)}.$$

Using to (32), we obtain

$$\log M(r,f) \le (1+o(1))\lambda(r) + \varepsilon \frac{\lambda(r)}{\log \mu(r)} + \frac{C_9}{\sqrt{\mu(r)}\log \mu(r)}, \quad r > r_0.$$

The relation (31) is proved.

Consider now the integral means $m_q(r,f)$. According to Jensen formula and Jensen inequality we have

$$N(r) = \frac{1}{2\pi q}\int_0^{2\pi} \log|f(re^{i\theta})|^q d\theta \le \frac{1}{q}\log\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^q d\theta\right) =$$

$$\log\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^q d\theta\right)^{1/q} = \log m_q(r,f) \le \log M(r,f), \qquad 1 \le q < +\infty.$$

The proof of Theorem is completed.

Remark. The existence of an entire function for which

$$m_2(r, \log|f|) \sim \lambda(r), \qquad r \to \infty,$$

in general outside an exceptional set follows also from the main theorem of [2].

1. *Clunie J. and Kövari J.* On integral functions having prescribed asymptotic growth, II.// Can. J. Math. – 1966. – P. 7–20.
2. *Girnyk M. and Gol'dberg A.* Approximation of subharmonic functions by logarithms of moduli of entire functions in integral metric// Isr.Math.Conf.Proc. – 2001. – **15.** – 135 p.
3. *Kondratyuk A. A.* Fourier series and meromorphic functions (in Russian)// Izdat. L'viv Univ., L'viv. – 1998.
4. *Miles J. B. and Shea D. F.* An extremal problem in value distribution theory// Quart. J. Math. Oxford. – 1973. – **24.** – P. 377–383.
5. *Pólya G., Szegö G.* Problems and theorems of analysis (in Russian)// Moscow. – 1978.
6. *Rubel L. A., Taylor B. A.* A Fourier series method for meromorphic and entire functions// Bull. Soc. Math. France. – 1968. – **96.** – P. 53–96.
7. *Vinogradov J.* Foundation of the numbers theory (in Russian)// Moscow. – 1949.

ЦІЛІ ФУНКЦІЇ ІЗ ЗАДАНИМ ЗРОСТАННЯМ ЇХ ОСНОВНИХ ХАРАКТЕРИСТИК

Побудована ціла функція f *із заданим зростанням її основних характеристик, а саме*

$$N(r,f_\mu)\sim T(r,f_\mu)\sim \log m_q(r,f_\mu)\sim \log M(r,f_\mu)\sim m_2(r,\log|f_\mu|)\sim\lambda(r),\quad r\to+\infty,$$

де m_q *—* q *-інтегральні середні,* $\lambda(r)$ *— додатна, неперервна, опукла відносно* $\log r$ *функція, яка зростає до* $+\infty$.

Така ціла функція будується у вигляді деякого нескінченного добутку.

ЦЕЛЫЕ ФУНКЦИИ С ЗАДАННЫМ РОСТОМ ИХ ХАРАКТЕРИСТИК

Построена целая функция f *с заданным ростом ее основных характеристик, а именно*

$$N(r,f_\mu)\sim T(r,f_\mu)\sim \log m_q(r,f_\mu)\sim \log M(r,f_\mu)\sim m_2(r,\log|f_\mu|)\sim\lambda(r),\quad r\to+\infty,$$

где m_q *—* q *-интегральные средние,* $\lambda(r)$ *— положительная, непрерывная, возрастающая к* $+\infty$ *выпуклая относительно* $\log r$ *функция.*

Такая целая функция строится в виде некоторого бесконечного произведения.

Львівський національний
університет ім. І. Франка

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