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ON SOME CLASS OF FACTORIZED OPERATOR DYNAMICAL SYSTEMS AND THEIR INTEGRABILITY

There is considered a problem of describing the corresponding factorized operator equations subject to a general factorized pseudo-differential symbol satisfying the standard Lax-type equation. The associated Poissonian structures are found by means of a special Backlund-type transform.

1. General setting. Consider a usual Tr-metrizable associative operator algebra $\mathfrak{g}$ endowed with the standard commutator Lie structure and admitting a decomposition into two Lie subalgebras $\mathfrak{g}_+ \oplus \mathfrak{g}_- = \mathfrak{g}$. Then due to the standard [4] Lie-algebraic theory of dynamical systems, one can construct on $\mathfrak{g}^*$ a so called Lax flow as follows:

$$\frac{d\ell}{dt} = [\nabla\gamma_+(\ell), \ell], \tag{1}$$

here $\ell \in \mathfrak{g}^*$, $\gamma \in I(\mathfrak{g}^*)$ is a Casimir function on $\mathfrak{g}^*$, that is $[\nabla\gamma(\ell), \ell] = 0$ with the associated gradient decomposition $\nabla\gamma(\ell) := \nabla\gamma_+(\ell) \oplus \nabla\gamma_-(\ell)$ for all $\ell \in \mathfrak{g}^*$ and $t \in \mathbb{R}$ being an evolutions parameter.

The flow (1) is called Lax-type integrable since all Casimir functions on $\mathfrak{g}^*$ generate invariants of (1) commuting to each other due to the well known Adler – Kostant – Symes theorem. In general a Casimir function $\gamma \in I(\mathfrak{g}^*)$ can be constructed as an analytical functional on $\mathfrak{g}^*$ in the following form:

$$\gamma := \mathrm{Tr}\,\gamma[\ell], \tag{2}$$

where, by definition, $\mathrm{Tr}(ab) := (a, b)$ is the mentioned above ad-invariant non-degenerate symmetric Tr-metrics on $\mathfrak{g} \simeq \mathfrak{g}^*$.

The expression (1) defines evidently the Hamiltonian vector field d/dt on $\mathfrak{g}^*$ with respect to the usual Lie – Poisson bracket on $\mathfrak{g}^*$ subject to the modified [3, 4] Lie-bracket $[\cdot,\cdot]_{\mathbb{R}} := [P_+(\cdot), P_+(.)] - [P_-(\cdot), P_-(\cdot)]$ on $\mathfrak{g}$, where $P_\pm\mathfrak{g} := \mathfrak{g}_\pm$ are the corresponding projectors. Take now another element $\tilde{\ell} \in \mathfrak{g}^*$ and construct the flow d/dt on $\mathfrak{g}^*$:

$$\frac{d\tilde{\ell}}{dt} = [\nabla\tilde{\gamma}_+(\tilde{\ell}), \tilde{\ell}], \tag{3}$$

where we assumed that $\tilde{\gamma} = \gamma \in I(\mathfrak{g}^*)$. So we have built two integrable flows (1) and (3) subject to the same Casimir function $\gamma \in I(\mathfrak{g}^*)$, generating the same vector field d/dt on $\mathfrak{g}^*$. Now we pose the following problem: find the relationships between elements ℓ and $\tilde{\ell} \in \mathfrak{g}^*$ evolving with respect to flows (1) and (3) and describe their dual Hamiltonian properties. This problem will be treated in detail below.

2. Factorization properties. Due to the Lax form of equations (1) and (3) there exist one-parametric subgroups $a(t)$ and $\tilde{a}(t) \in \exp \mathfrak{g}_+$, $t \in \mathbb{R}$, such that for any $\ell(0)$ and $\tilde{\ell}(0) \in \mathfrak{g}^*$

$$\ell(t) = Ad^*_{a(t)}\ell(0) = a^{-1}(t)\ell(0)a(t)\,,$$

$$\tilde{\ell}(t) = Ad^*_{\tilde{a}(t)}\ell(0) = \tilde{a}^{-1}(t)\tilde{\ell}(0)\tilde{a}(t)\,, \tag{4}$$

where evidently

$$\frac{da(t)}{dt} = -\,a(t)\nabla\gamma_+(\ell), \qquad \frac{d\tilde{a}(t)}{dt} = -\,\tilde{a}(t)\nabla\gamma_+(\tilde{\ell}) \tag{5}$$

for all $t \in \mathbb{R}$. From (4) one gets easily that for all $t \in \mathbb{R}$

$$a(t)\ell(t)a^{-1}(t) = \ell(0), \qquad \tilde{a}(t)\tilde{\ell}(t)\tilde{a}^{-1}(t) = \tilde{\ell}(0). \tag{6}$$

Assume now that there exists an element $B(0) \in \exp \mathfrak{g}_+$ such that the expression

$$Ad^*_{B(0)}\ell(0) = \tilde{\ell}(0)$$

holds, or equivalently

$$B^{-1}(0)\ell(0)B(0) = \tilde{\ell}(0). \tag{7}$$

Whence the equalities (6) give rise to the following relationships:

$$\tilde{\ell} = B^{-1}\ell B\,, \tag{8}$$

where, by definition, $B \in \exp \mathfrak{g}_+$ is given as

$$B := B(t) = a^{-1}(t)B(0)\tilde{a}(t) \tag{9}$$

for all $t \in \mathbb{R}$.

Let us now assume that an element $A \in \exp \mathfrak{g}_+$ being defined as

$$A := \ell B\,. \tag{10}$$

This is equivalent, evidently, to the statement that the expression $A(0) = \ell(0)B(0) \in \exp \mathfrak{g}_+$ holds for the element $\ell(0) \in \mathfrak{g}^*$. As a result of the representations (10) and (9) one finds easily the following evolution equations on $A, B \in \exp \mathfrak{g}_+$ written first in [2]:

$$\frac{dA}{dt} = \nabla\gamma_+(\ell)A - A\nabla\gamma_+(\tilde{\ell}), \qquad \frac{dB}{dt} = \nabla\gamma_+(\ell)B - B\nabla\gamma_+(\tilde{\ell}) \tag{11}$$

for all $t \in \mathbb{R}$. Thereby we have stated the following factorizing representation theorem.

Theorem 1. *Let an element* $\ell \in \mathfrak{g}^*$ *is factorization* $\ell = AB^{-1}$ *with* $A, B \in \exp \mathfrak{g}_+$. *Then the Lax-type flows* (1) *and* (3) *are factorized too into two flows* (11) *with the element* $\tilde{\ell} = B^{-1}A = A^{-1}\ell A \in \mathfrak{g}^*$.

A p r o o f is needed only for the last representation $\tilde{\ell} = B^{-1}A = A^{-1}\ell A \in \mathfrak{g}^*$. Since owing to (8) $\tilde{\ell} = B^{-1}\ell B$, from (10) one gets right away that $\tilde{\ell} = B^{-1}A = IB^{-1}A \equiv A^{-1}\ell B \cdot B^{-1}A = A^{-1}\ell A \in \mathfrak{g}^*$, that ends the proof. ◊

Thus, we constructed two factorized equations (11) subject to the representations $\ell = AB^{-1}$, $\tilde{\ell} = B^{-1}A \in \mathfrak{g}^*$ with elements $A, B \in \exp \mathfrak{g}_+$ and the common invariant Casimir function $\gamma(\ell) = \gamma(\tilde{\ell}) \in I(\mathfrak{g}^*)$. Below we proceed to analyzing Hamiltonian properties of obtained above flows (11).

3. Hamiltonian analysis. Let us consider flows (1) and (2) as being Hamiltonian on $\mathfrak{g}^* \oplus \mathfrak{g}^*$ subject to the following tonsorial doubled standard Poissonian structure:

$$\vartheta : \begin{pmatrix} \nabla\gamma(\ell) \\ \nabla\gamma(\ell^*) \end{pmatrix} \longrightarrow \begin{pmatrix} [\nabla\gamma_+(\ell), \ell\,] - [\nabla\gamma(\ell), \ell\,]_+ \\ [\nabla\gamma_+(\tilde{\ell}), \tilde{\ell}\,] - [\nabla\gamma(\tilde{\ell}), \tilde{\ell}\,]_+ \end{pmatrix}, \tag{12}$$

where $\gamma \in D(\mathbf{g}^*)$ is any smooth functional on $\mathbf{g}^* \oplus \mathbf{g}^*$. Concerning the transformation

$$\Phi(A,B;\ell,\tilde{\ell}) = 0 \;\Leftrightarrow\; \ell - AB^{-1} = 0, \qquad \tilde{\ell} - B^{-1}A = 0\,, \tag{13}$$

which can be evidently considered as a usual Backlund transformation, we can construct a new Poisson structure $\eta : T^*(\mathbb{G}_+ \times \mathbb{G}_+) \to T(\mathbb{G}_+ \times \mathbb{G}_+)$ on the subgroup space $\mathbb{G}_+ \times \mathbb{G}_+$ with respect to the phase variables $(A,B) \in \mathbb{G}_+ \times \mathbb{G}_+$. Thereby one finds [3, 4] the corresponding to (12) and (13) transformed Poissonian structure $\eta : T^*(\mathbb{G}_+ \times \mathbb{G}_+) \to T(\mathbb{G}_+ \times \mathbb{G}_+)$ at $(A,B) \in \mathbb{G}_+ \times \mathbb{G}_+$, where

$$\eta = T\vartheta T^*,$$

$$T = \Phi'_{(\ell,\tilde{\ell})}\,(\Phi'_{(A,B)})^{-1}. \tag{14}$$

Making use of the expressions

$$\Phi'_{(A,B)} = \begin{pmatrix} -(\cdot)B^{-1} & \ell(\cdot)B^{-1} \\ -B^{-1}(\cdot) & B^{-1}(\cdot)\tilde{\ell} \end{pmatrix}, \qquad \Phi'_{(\ell,\tilde{\ell})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(\Phi'_{(A,B)})^{-1} = \begin{pmatrix} -(1-\ell\otimes\tilde{\ell}^{-1})^{-1}(\cdot)B & (1-\ell\otimes\tilde{\ell}^{-1})^{-1}\ell B(\cdot)\tilde{\ell}^{-1} \\ -(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}(\cdot)B & (1-\tilde{\ell}^{-1}\otimes\ell)^{-1}B(\cdot) \end{pmatrix},$$

$$(\Phi'^{*}_{(A,B)})^{-1} = \begin{pmatrix} -B(\cdot)(1-\tilde{\ell}^{-1}\otimes\ell\,)^{-1} & -B(\cdot)(1-\tilde{\ell}^{-1}\otimes\ell\,)^{-1} \\ \tilde{\ell}^{-1}(\cdot)B\ell(1-\tilde{\ell}^{-1}\otimes\ell)^{-1} & (\cdot)B(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}\,\ell \end{pmatrix}, \tag{15}$$

jointly with the ϑ-structure (12), one gets from (14) that

$$\eta = \eta_1 \cdot \eta_2, \qquad \eta_2 = \begin{pmatrix} \eta_2^{11} & \eta_2^{12} \\ \eta_2^{21} & \eta_2^{22} \end{pmatrix},$$

$$\eta_1 = \begin{pmatrix} -(1-\ell\otimes\tilde{\ell}^{-1})^{-1}(\cdot)B & (1-\ell\otimes\tilde{\ell}^{-1})^{-1}\ell B(\cdot)\tilde{\ell}^{-1} \\ -(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}(\cdot)B & (1-\tilde{\ell}^{-1}\otimes\ell)^{-1}B(\cdot) \end{pmatrix},$$

$$\eta_2^{11} = \big[\ell,\big((1-\ell\otimes\tilde{\ell}^{-1})^{-1}(\cdot)B(1-\ell\otimes\tilde{\ell}^{-1})^{-1}B(.)\big)_+\big] -$$

$$-\big[\tilde{\ell}^{-1}(\cdot)B\ell(1-\tilde{\ell}^{-1}\otimes\ell)^{-1},\tilde{\ell}\,\big] + \big[\big((\cdot)B(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}\big)_+,\tilde{\ell}\,\big],$$

$$\eta_2^{12} = -\big[\big(B(\cdot)(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}\big)_+,\ell\big] + \big[\big(B(\cdot)(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}\big),\ell\big]_+$$

$$\eta_2^{21} = \big[\big(\tilde{\ell}^{-1}(\cdot)B\ell(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}\big)_+,\tilde{\ell}\,\big] - \big[\tilde{\ell}^{-1}(\cdot)B\ell(1-\tilde{\ell}^{-1}\otimes\ell)^{-1},\tilde{\ell}\,\big] -$$

$$-\big[\ell,(1-\ell\otimes\tilde{\ell}^{-1})^{-1}(\cdot)B(1-\ell\otimes\tilde{\ell}^{-1})^{-1}(\cdot)\big]_+,$$

$$\eta_2^{22} = \big[\big((\cdot)B(1-\tilde{\ell}^{-1}\otimes\ell)^{-1}\big)_+,\tilde{\ell}\big] - \big[(\cdot)B(1-\tilde{\ell}^{-1}\otimes\ell)^{-1},\tilde{\ell}\big]_+ \tag{16}$$

at $\ell = AB^{-1}$ and $\tilde{\ell} = B^{-1}A \in \mathbf{g}^*$.

Let now take any functional $\gamma \in I(\mathfrak{g}^*)$ and construct the functional $\tilde{\gamma} := \gamma_{\ell = AB^{-1}} \in D(\mathbb{G}_+ \times \mathbb{G}_+)$. Then one constructs due to the Poissonian bracket (16) the following Hamiltonian flow on $\mathbb{G}_+ \times \mathbb{G}_+$:

$$\frac{d}{d\tau}(A, B)^{\top} = \eta \nabla \tilde{\gamma}(A, B), \qquad (17)$$

where $(A, B) \in \mathbb{G}_+ \times \mathbb{G}_+$ and $\tau \in \mathbb{R}$ is an evolution parameter. The flow (17) is characterized by the following

Theorem 2. *The Hamiltonian vector field* $d/d\tau$ *on* $\mathbb{G}_+ \times \mathbb{G}_+$, *defined by* (17), *and the vector field* d/dt, *defined by* (11), *coincide on* $\mathbb{G}_+ \times \mathbb{G}_+$.

P r o o f of this theorem consists in simple but a bit tedious calculation of the expression (17). ◊

The result above solves completely a problem posed in [2] about Hamiltonian formulation of the factorized equations (11).

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ПРО ДЕЯКІ КЛАСИ ФАКТОРИЗОВАНИХ ДИНАМІЧНИХ СИСТЕМ І ЇХ ІНТЕГРОВНІСТЬ

Розглядається задача опису факторизованих операторних рівнянь, відповідних до загального факторизованого псевдодиференціального символу, що задовольняє стандартне рівняння типу Лакса. Асоційовані структури Пуассона визначено за допомогою спеціального перетворення типу Беклунда.

О НЕКОТОРЫХ КЛАССАХ ФАКТОРИЗОВАННЫХ ДИНАМИЧЕСКИХ СИСТЕМ И ИХ ИНТЕГРИРУЕМОСТИ

Рассматривается задача описания факторизованных операторных уравнений, соответствующих общему факторизованному псевдодиффференциальному символу, удовлетворяющему стандартному уравнению типа Лакса. Ассоциированные структуры Пуассона определены с помощью специального преобразования типа Беклунда.

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