UDC 512.536 + 512.568.2

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ON FEEBLY COMPACT SEMITOPOLOGICAL SEMILATTICE $\exp_n \lambda$

We study feebly compact shift-continous topologies on the semilattice $(\exp_n \lambda, \mathbf{I})$. *It is proved that such T_1-topology is sequentially pracompact if and only if it is* $D(\omega)$ *-compact.*

Key words and phrases: *semitopological semilattice, feebly compact, H-closed, infra H-closed, Y-compact, sequentially countably pracompact, selectively sequentially feebly compact, selectively feebly compact, sequentially feebly compact, the Sunflower Lemma, Δ-system.*

We shall follow the terminology of [4, 9, 10, 23]. If X is a topological space and $A \subseteq X$, then by $cl_X(A)$ and $\operatorname{int}_X(A)$ we denote the closure and the interior of A in X, respectively. By ω we denote the first infinite cardinal and by $\mathbb{N}$ the set of positive integers. By $D(\omega)$ and $\mathbb{R}$ we denote an infinite countable discrete space and the real numbers with the usual topology, respectively.

A subset A of a topological space X is called *regular open* if $\operatorname{int}_X(\operatorname{cl}_X(A)) = A$.

We recall that a topological space X is said to be

- *semiregular,* if X has a base consisting of regular open subsets;
- *compact,* if each open cover of X has a finite subcover;
- *sequentially compact,* if each sequence of points of X has a convergent subsequence in X;
- *countably compact,* if each open countable cover of X has a finite subcover;
- *H-closed,* if X is a closed subspace of every Hausdorff topological space in which it is contained;
- *infra H-closed,* if any continuous image of X into any first countable Hausdorff space is closed (see [18]);
- *totally countably pracompact,* if there exists a dense subset D of the space X such that each sequence of points of the set D has a subsequence with the compact closure in X;
- *sequentially pracompact,* if there exists a dense subset D of the space X such that each sequence of points of the set D has a convergent subsequence ([15]);
- *countably compact at a subset $A \subseteq X$*, if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact,* if there exists a dense subset A of X such that X is countably compact at A;
- *selectively sequentially feebly compact,* if for every family $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence [7];

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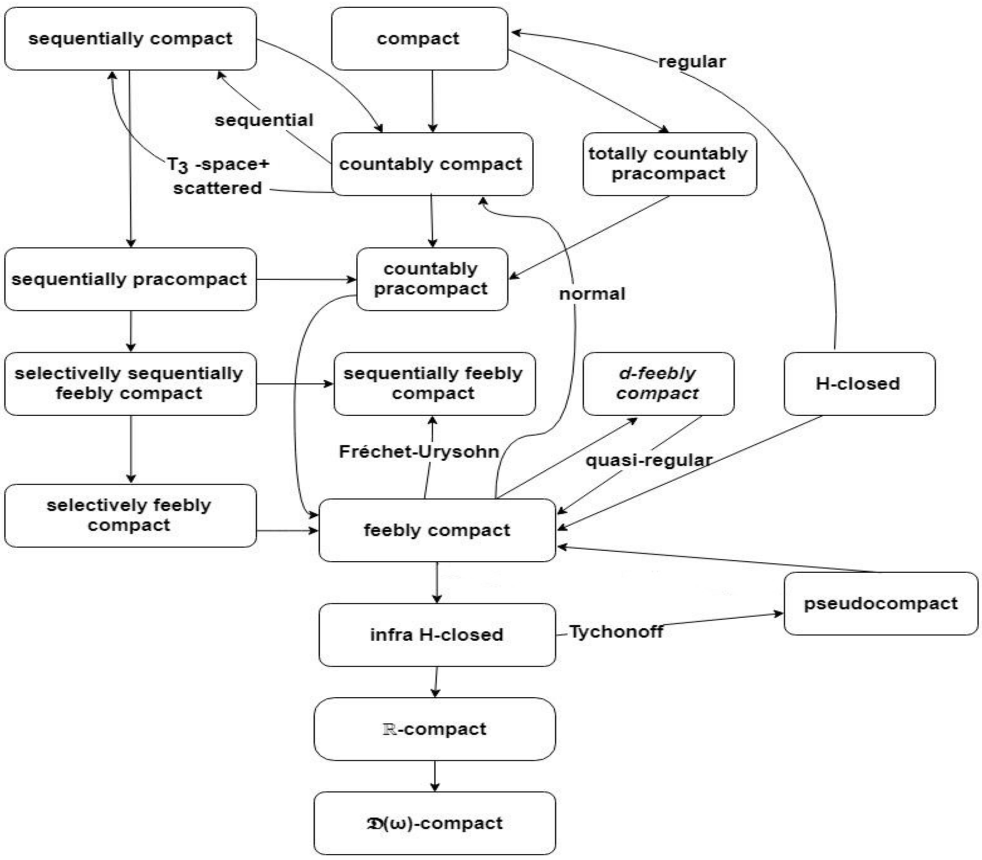
- *sequentially feebly compact,* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \varnothing\}$ is finite for every open neighborhood W of x, see [8];
- *selectively feebly compact,* for each sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x \in X$ and a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : x_n \in W\}$ is infinite for every open neighborhood W of x [7];
- *feebly compact* if each locally finite open cover of X is finite [3];
- *d-feebly compact* (or *DFCC*), if every discrete family of open subsets in X is finite [21];
- *pseudocompact,* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *Y-compact* (for some topological space Y), if for any continuous map $f : X \to Y$ a set $f(X)$ is compact.

The following diagram describes relations between the above defined classes of topological spaces.



A *semilattice* is a commutative semigroup of idempotents. On a semilattice S there exists a natural partial order, namely, for any elements e and f of S $e \leq f$ if and only if $ef = fe = e$. For any element e of a semilattice S we put

$$\uparrow e = \{f \in S : e \leq f\}.$$

A *topological* (*semitopological*) *semilattice* is a topological space together with a continuous (separately continuous) semilattice operation. If S is a semilattice and τ is a topology on S such that (S,τ) is a topological semilattice, then we shall call τ a *semilattice topology* on S, and if τ is a topology on S such that (S,τ) is a semitopological semilattice, then we shall call τ a *shift-continuous topology* on S.

For an arbitrary positive integer n and an arbitrary non-zero cardinal λ we put

$$\exp_n \lambda = \{A \subseteq \lambda : |A| \le n\}.$$

It is obvious that for any positive integer n and any non-zero cardinal λ the set $\exp_n \lambda$ with the binary operation $\cap$ is a semilattice. Later in this paper by $\exp_n \lambda$ we shall denote the semilattice $(\exp_n \lambda; \cap)$.

This paper is a continuation of [16] and [17]. In [16] we studied feebly compact semitopological semilattices $\exp_n \lambda$. Therein, all compact T_1-topological semilattices $\exp_n \lambda$ were described. In [16] it was proved that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every T_1-semitopological countably compact semilattice $\exp_n \lambda$ is compact topological semilattice. Also, there we constructed a countably pracompact H-closed quasiregular non-semiregular topology τ_{fc}^2 such that $\left(\exp_2 \lambda, \tau_{fc}^2\right)$ is a semitopological semilattice with the discontinuous semilattice operation and show that for arbitrary positive integer n and arbitrary infinite cardinal λ a semiregular feebly compact semitopological semilattice $\exp_n \lambda$ is a compact topological semilattice. In [17] we proved that for any shift-continuous T_1-topology τ on $\exp_n \lambda$ the following conditions are equivalent: *(i)* τ is countably pracompact; (*ii*) τ is feebly compact; (*iii*) τ is d-feebly compact; (*iv*) $(\exp_n \lambda, \tau)$ is an H-closed space.

In [2] was proved that every pseudocompact topological group is sequentially feebly compact. Also, by Corollary 4.6 of [7], the Cantor cube $D^{\mathfrak{c}}$ is selectively sequentially feebly compact. By [9, Theorem 3.10.33] $D^{\mathfrak{c}}$ is not sequentially compact. Therefore, the compact topological group $G = D^{\mathfrak{c}}$ is selectively sequentially feebly compact but not sequentially feebly compact. Also, there exists a dense subgroup of $\mathbb{Z}_2^{\mathfrak{c}}$, where $\mathbb{Z}_2^{\mathfrak{c}}$ is the $\mathfrak{c}$-th Tychonoff power of the discrete cyclic two-element group, which is selectively pseudocompact but not selectively sequentially pseudocompact [24]. This and our above results of [16] and [17] motivates us to establish selective (sequential) feeble compactness of the semilattice $\exp_n \lambda$ as a semitopological semigroup.

Namely, we show that a T_1-semitopological semilattice $\exp_n \lambda$ is sequentially countably pracompact if and only if it is $\mathfrak{D}(\omega)$-compact.

Lemma 1. *Let n be any positive integer and λ be any infinite cardinal. Then the set of isolated points of a T_1-semitopological semilattice $\exp_n \lambda$ is dense in it.*

P r o o f. Fix an arbitrary non-empty open subset U of $\exp_n \lambda$. There exists $y \in \exp_n \lambda$ such that ${\uparrow} y \cap U = \{y\}$. By Proposition 1(*iii*) from [16], ${\uparrow} y$ is an open-and-closed subset of $\exp_n \lambda$, so y is an isolated point of $\exp_n \lambda$. ♦

A family of non-empty sets $\{A_i : i \in \mathrm{I}\}$ is called a Δ-*system* (*a sunflower* or *a* Δ-*family*) if the pairwise intersections of the members is the same, i.e., $A_i \cap A_j = S$ for some set S (for $i \neq j$ in I) [20]. The following statement is well known as the *Subnlower Lemma* or the *Lemma about a* Δ-*system* (see [20, p. 107]).

Lemma 2. *Every infinite family of* n*-element sets* ($n < \omega$) *includes an infinite* Δ*-subfamily.*

Proposition 1. Let n *be any positive integer and* λ *be any infinite cardinal. Then every feebly compact* T_1*-semitopological semilattice* $\exp_n \lambda$ *is sequentially pracompact.*

P r o o f. Suppose to the contrary that there exists a feebly compact T_1-semitopological semilattice $\exp_n \lambda$ which is not sequentially pracompact. Then every dense subset D of $\exp_n \lambda$ contains a sequence of points from D which has not a convergent subsequence.

By Proposition 1 of [17] the subset $\exp_n \lambda \setminus \exp_{n-1} \lambda$ is dense in $\exp_n \lambda$ and by Proposition 1 (*ii*) of [16] every point of the set $\exp_n \lambda \setminus \exp_{n-1} \lambda$ is isolated in $\exp_n \lambda$ Then the set $\exp_n \lambda \setminus \exp_{n-1} \lambda$ contains an infinite sequence of points $\{x_p : p \in ¥\}$ which has not a convergent subsequence. By Lemma 2 the sequence $\{x_p : p \in ¥\}$ contains an infinite Δ-subfamily, that is an infinite subsequence $\{x_{p_i} : i \in ¥\}$ such that there exist $x \in \exp_n \lambda$ such that $x_{p_i} \cap x_{p_j} = x$ for any distinct $i, j \in ¥$.

Suppose that $x = 0$ is zero of the semilattice $\exp_n \lambda$. Since the sequence $\{x_{p_i} : i \in ¥\}$ is an infinite Δ-subfamily, the intersection $\{x_{p_i} : i \in ¥\} \cap \uparrow y$ contains at most one set for every non-zero element $y \in \exp_n \lambda$. Thus $\exp_n \lambda$ contains an infinite locally finite family of open non-empty subsets which contradicts the feeble compactness of $\exp_n \lambda$.

If x is a non-zero element of the semilattice $\exp_n \lambda$ then by Proposition 1 (*ii*) of [16], $\uparrow x$ is an open-and-closed subspace of $\exp_n \lambda$, and hence by Theorem 14 from [3] the space $\uparrow x$ is feebly compact. We observe that x is zero of the semilattice $\uparrow x$ which contradicts so similarly the previous part of the proof. We obtain a contradiction. ♦

Proposition 2. Let n *be any positive integer and* λ *be any infinite cardinal. Then every feebly compac* T_1*-semitopological semilattice* $\exp_n \lambda$ *is totally countably pracompact.*

P r o o f. We put $D = \exp_n \lambda \setminus \exp_{n-1} \lambda$. By Proposition1 of [17] the subset D is dense in $\exp_n \lambda$ and by Proposition 1 (*ii*) of [16] every point of the set D is isolated in $\exp_n \lambda$. Fix an arbitrary sequence $\{x_p : p \in ¥\}$ of points of D. By Lemma 2 the sequence $\{x_p : p \in ¥\}$ contains an infinite Δ-subfamily.

Suppose that $x = 0$ is the zero of the semilattice $\exp_n \lambda$. Since the sequence $\{x_{p_i} : i \in ¥\}$ is an infinite Δ-subfamily, the intersection

$\{x_{p_i} : i \in \mathbb{N}\} \cap \uparrow y$ contains at most one point of the sequence for every non-zero element $y \in \exp_n \lambda$. By Proposition 1 (*ii*) of [16] for every point $a \in \exp_n \lambda \setminus \{0\}$ there exists an open neighbourhood $U(a)$ of a in $\exp_n \lambda$ such that $U(a) \subseteq \uparrow a$ and hence our assumption implies that zero 0 is a unique accumulation point of the sequence $\{x_{p_i} : i \in \mathbb{N}\}$. Since by Lemma 1 from [16] for an arbitrary open neighbourhood $W(0)$ of zero 0 in $\exp_n \lambda$ there exist finitely many non-zero elements $y_1, \ldots, y_k \in \exp_n \lambda$ such that

$$(\exp_n \lambda \ , \ \exp_{n-1} \lambda) \subseteq W(0) \cup \uparrow y_1 \cup \ldots \cup \uparrow y_k ,$$

we get that $\mathrm{cl}_{\exp_n \lambda}(\{x_{p_i} : i \in \mathbb{N}\}) = \{0\} \cup \{x_{p_i} : i \in \mathbb{N}\}$ is a compact subset of $\exp_n \lambda$.

If x is a non-zero element of the semilattice $\exp_n \lambda$ then by Proposition 1 (*ii*) of [16], $\uparrow x$ is an open-and-closed subspace of $\exp_n \lambda$, and hence by Theorem 14 of [3] the space $\uparrow x$ is feebly compact. Then x is zero of the semilattice $\uparrow x$ and by the previous part of the proof we have that $\mathrm{cl}_{\exp_n \lambda}(\{x_{p_i} : i \in \mathbb{N}\}) = \{x\} \cup \{x_{p_i} : i \in \mathbb{N}\}$ is a compact subset of the semilattice $\exp_n \lambda$. ♦

Proposition 3. Let n be any positive integer and λ be any infinite cardinal. Then every $\mathfrak{D}(\omega)$-compact T_1-semitopological semilattice $\exp_n \lambda$ is feebly compact.

P r o o f. Suppose to the contrary that there exist a $\mathfrak{D}(\omega)$-compact T_1-semitopological semilattice $\exp_n \lambda$ which is not feebly compact. Then there exists an infinite locally finite family $\mathcal{U} = \{U_i\}$ of open non-empty subsets of $\exp_n \lambda$. Without loss of generality we may assume that the family $\mathcal{U} = \{U_i\}$ is countable, i.e., $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$. Lemma 1 implies that for every $U_i \in \mathcal{U}$ there exists $a_i \in U_i$ such that $\mathcal{U}^* = \{\{a_i\} : i \in \mathbb{N}\}$ is a family of isolated points of $\exp_n \lambda$. Since the family $\mathcal{U}$ is locally finite, without loss of generality we may assume that $a_i \neq a_j$ for distinct $i, j \in \mathbb{N}$. The family $\mathcal{U}^*$ is locally finite as refinement of a locally finite familly $\mathcal{U}$. Since $\exp_n \lambda$ is a T_1-space, $\bigcup \mathcal{U}^*$ is a closed subset in $\exp_n \lambda$ and hence the map $f : \exp_n \lambda \to \mathbb{N}_{\mathrm{d}}$, where $\mathbb{N}_{\mathrm{d}}$ is the set of positive integers with the discrete topology, defined by the formula

$$f(b) = \begin{cases} 1, & \text{if } b \in \exp_n \lambda \setminus \bigcup \mathcal{U}^*; \\ i+1, & \text{if } b = a_i \text{ for some } i \in \mathbb{N}, \end{cases}$$

is continuous. This contradicts $\mathfrak{D}(\omega)$-compactness of the space $\exp_n \lambda$, because every two infinite countable discrete spaces are homeomorphic. ♦

We summarise our results in the following theorem.

Theorem 1. *Let n be any positive integer and λ be any infinite cardinal. Then for any T_1-semitopological semilattice $\exp_n \lambda$ the following conditions are equivalent:*

(i) $\exp_n \lambda$ *is sequentially pracompact;*

(ii) $\exp_n \lambda$ *is totally countably pracompact;*

(iii) $\exp_n \lambda$ *is feebly compact;*

(iv) $\exp_n \lambda$ *is* $\mathfrak{D}(\omega)$*-compact.*

P r o o f. Implications $(i) \Rightarrow (iii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are trivial. The corresponding their converse implications $(iii) \Rightarrow (i)$, $(iii) \Rightarrow (ii)$ and $(iv) \Rightarrow (iii)$ follow from Propositions 1, 2 and 3, respectively. ♦

It is well known that the (Tychonoff) product of pseudocompact spaces is not pseudocompact (see [9, Section 3.10]. On the other hand Comfort and Ross in [6] proved that a Tychonoff product of an arbitrary non-empty family of pseudocompact topological groups is a pseudocompact topological group. Ravsky in [22] generalized Comfort-Ross Theorem and proved that a Tychonoff product of an arbitrary non-empty family of feebly compact paratopological groups is feebly compact. Also, a counterpart of the Comfort-Ross Theorem for pseudocompact primitive topological inverse semigroups and primitive inverse semiregular feebly compact semitopological semigroups with closed maximal subgroups were proved in [11] and [14], respectively.

Since a Tychonoff product of *H*-closed spaces is *H*-closed (see [5], Theorem 3] or [9, 3.12.5 (*d*)]) Theorem 1 implies a counterpart of the Comfort-Ross Theorem for feebly compact semitopological semilattices $\exp_n \lambda$:

Corollary 1. *Let* $\left\{\exp_{n_i} \lambda_i : i \in I\right\}$ *be a family of non-empty feebly compact* T_1*-semitopological semilattices and* $n_i \in \mathbb{N}$ *for all* $i \in I$*. Then the Tychonoff product* $\prod\left\{\exp_{n_i} \lambda_i : i \in I\right\}$ *is feebly compact.*

Definition 1. If $\{X_i : i \in I\}$ is a family of sets, $X = \prod\{X_i : i \in I\}$ is their Cartesian product and p is a point in X, then the subset

$$\Sigma(p, X) = \left\{x \in X : \left|\{i \in I : x(i) \neq p(i)\}\right| \leq \omega\right\}$$

of X is called the Σ*-product* of $\{X_i : i \in I\}$ with the basis point $p \in X$. In the case when $\{X_i : i \in I\}$ is a family of topological spaces we already assume that $\Sigma(p, X)$ is a subspace of the Tychonoff product $X = \prod\{X_i : i \in I\}$.

It is obvious that if $\{X_i : i \in I\}$ is a family of semilattices then $X = \prod\{X_i : i \in I\}$ is a semilattice as well. Moreover $\Sigma(p, X)$ is a subsemilattice of X for arbitrary $p \in X$. Then Theorem 1 and Proposition 2.2 of [15] imply the following corollary.

Corollary 2. *Let* $\left\{\exp_{n_i} \lambda_i : i \in I\right\}$ *be a family of non-empty feebly compact* T_1*-semitopological semilattices and* $n_i \in \mathbb{N}$ *for all* $i \in I$*. Then for every point* p *of the product* $X = \prod\left\{\exp_{n_i} \lambda_i : i \in I\right\}$ *the* Σ*-product* $\Sigma(p, X)$ *is feebly compact.*

Acknowledgements. The authors acknowledge Alex Ravsky and the referee for their comments and suggestions.

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DOI: 10.3390/axioms7040086

ПРО СЛАБКО КОМПАКТНУ НАПІВТОПОЛОГІЧНУ НАПІВГРАТКУ $\exp_n \lambda$

В статті досліджуються трансляційно неперервні топології τ *на напівгратці* $(\exp_n \lambda; \mathbf{I})$. *Доведено, що трансляційно неперервна* T_1*-топологія* τ *на* $(\exp_n \lambda; \mathbf{I})$ *є секвенціально пракомпактною тоді і лише тоді, коли вона є* $D(\omega)$*-компактною.*

Ключові слова: *напівтопологічні напівгратка, слабко компактний, H-замкнений, інфра H-замкнений, Y-компактний, секвенціально зліченно пракомпактний, селективно секвенціально слабко компактний, селективно слабко компактний, секвенціально слабко компактний, лема про соняшник,* Δ*-система.*

О СЛАБО КОМПАКТНОЙ ПОЛУТОПОЛОГИЧЕСКОЙ ПУЛУРЕШЁТКЕ $\exp_n \lambda$

В статье исследуются трансляционно непрерывные топологии τ *на полурешётке* $(\exp_n \lambda; \mathbf{I})$. *Доказано, что трансляционно непрерывная* T_1*-топология* τ *на* $(\exp_n \lambda; \mathbf{I})$, *секвенциально пракомпактна только тогда, когда она* $D(\omega)$*-компактна.*

Ключевые слова: *полутопологическая полурешетка, слабо компактный, H-замкнутый, инфра H-замкнутый, Y-компактный, секвенциально счетно пракомпактный, селективно секвенциально слабо компактный, селективно слабо компактный, секвенциально слабо компактный, лемма о подсолнухе,* Δ*-система.*

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Одержано
28.04.18