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ON THE SMITH NORMAL FORM OF LEAST COMMON MULTIPLE OF MATRICES FROM SOME CLASS OF MATRICES

For nonsingular matrices of arbitrary order over a commutative principal ideal domain under some restrictions on their canonical diagonal forms, the Smith normal form of their least common right multiple is constructed.

Key words: *principal ideal domain, Smith normal form, least common right multiple of matrices.*

Let R be a commutative principal ideal domain with $1_R \neq 0$ and let A be a $n \times n$ nonsingular matrix over R. Then there are invertible matrices P, Q, such that

$$PAQ = \mathrm{E} = \mathrm{diag}\,(\varepsilon_1, \dots, \varepsilon_n), \qquad \varepsilon_i \mid \varepsilon_{i+1}, \qquad i = 1, \dots, n-1 .$$

The matrix E is called the Smith normal form or canonical diagonal form of matrix A. Matrices P and Q are left and right transforming matrices of A, respectively.

If $A = BC$, then we say that B is a left divisor of matrix A and A is a right multiple of B. Moreover, if $M = AA_1 = BB_1$, then the matrix M is called a common right multiple of matrices A and B. If in addition the matrix M is a left divisor of any other common right multiple of matrices A and B, then we say that M is a least common right multiple of A and B ($[A, B]_r$ in notation).

The notion of the greatest common divisor (g.c.d.) and the least common multiple (l.c.m.) of matrices is one of the basic concepts of arithmetic of matrices over rings. They are widely used for the solution of applied problems of matrix algebra, in particular, in the problems of finding solutions of systems of unilateral matrix equations over fields. According to the generalized Bezout theorem, solving a system of unilateral matrix equations over fields can be reduced to finding common left linear unital divisors of the corresponding matrices. Since the common divisors are the divisors of the greatest common divisor, the study of g.c.d. and l.c.m. both for polynomial matrices and for matrices over wider classes of rings are relevant. We note that the ring of polynomial matrices is the ring of matrices over the commutative principal ideal domain.

The purpose of this article is to study the structure of the Smith normal form of the least common right multiple for nonsingular matrices of arbitrary order over the commutative principal ideal domain under some restrictions on their canonical diagonal forms. The study of the properties of the least common multiple over Euclidean rings and over rings of principal ideals is quite thorough, and only fragmentary in the case of rings of matrices. Among them we note the works V. Nanda [7], C. Yang, B. Li [10], S. Damkaew, S. Prugsapitak [4], N. Erawaty, M. Bahri, L. Haryanto, A. Amir [5], A. Romaniv, V. Shchedryk [1, 2], etc.

Suppose that p is an indecomposable element of the ring R. Let $R_{(p)}$ denote the localization of the ring R at the prime ideal (p). In other words, $R_{(p)}$ is the ring consisting of elements of the form $x = p^v \dfrac{a}{b}$, where a and b are elements of the ring R that are coprime to p and $v \in \mathbb{N} \cup \{0\}$.

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Note that the Smith normal form E of a nonsingular matrix $A \in M_n(R)$ can be written in the form:

$$\mathrm{E} = I\varepsilon_1 \operatorname{diag}\left(1, \frac{\varepsilon_2}{\varepsilon_1}, \dots, \frac{\varepsilon_2}{\varepsilon_1}\right) \times \operatorname{diag}\left(1, 1, \frac{\varepsilon_3}{\varepsilon_2}, \dots, \frac{\varepsilon_3}{\varepsilon_2}\right) \times \dots \times \operatorname{diag}\left(1, \dots, 1, \frac{\varepsilon_n}{\varepsilon_{n-1}}\right),$$

where I is identity $n \times n$-matrix. It follows that the matrix E can be written as the product of matrices of the form

$$\operatorname{diag}(1, \dots 1, p^{\mu}, \dots, p^{\mu}).$$

By theorem 5.6 from [6], the matrix A in the ring $R_{(p)}$ will be considered with this Smith normal form $\operatorname{diag}(1, \dots 1, p^{\mu}, \dots, p^{\mu})$. Therefore, it is advisable to consider this Smith normal form.

Let a nonsingular matrix $A \in M_n(R)$ has the Smith normal form

$$\mathrm{E} = \operatorname{diag}(\underbrace{1, \dots, 1}_{k}, \underbrace{\varepsilon, \dots, \varepsilon}_{n-k}) = P_A A Q_A, \qquad k = 1, \dots, n-1.$$

Suppose that the greatest common divisor of the minor of order $n-1$ of matrix B equals 1. Then

$$P_B B Q_B = \Delta, \qquad \Delta = \operatorname{diag}(1, \dots, 1, \delta).$$

By $\mathbf{P}_A$ and $\mathbf{P}_B$ we denote the sets of all left transforming matrices of matrices A and B, respectively. According to [8], $\mathbf{P}_A = \mathbf{G}_{\mathrm{E}} P_A$, $\mathbf{P}_B = \mathbf{G}_{\Delta} P_B$, where

$$\mathbf{G}_{\mathrm{E}} = \{H \in GL_n(R) \mid H\mathrm{E} = \mathrm{E}H_1,\, H_1 \in GL_n(R)\},$$

$$\mathbf{G}_{\Delta} = \{L \in GL_n(R) \mid \mathrm{L}\Delta = \Delta L_1,\, L_1 \in GL_n(R)\}.$$

Note that $\mathbf{G}_{\mathrm{E}}$ and $\mathbf{G}_{\Delta}$ are multiplicative groups.

We denote by (a, b) and $[a, b]$ the greatest common divisor and the least common multiple of the elements a and b, respectively. The notation $a \mid b$ means that the element a divides the element b.

Lemma 1. *Let* $P_B P_A^{-1} = \|s_{ij}\|_1^n = S$. *Then the element* $((\varepsilon, \delta), s_{n1}, \dots, s_{nk})$, $k = 1, \dots, n-1$, *where* $\delta \in R$ *is the last invariant factor of the Smith normal form* Δ *of the matrix* B, *and it is an invariant with respect to transforming matrices from* $\mathbf{P}_A$ *and* $\mathbf{P}_B$.

P r o o f. Let $F_A \in \mathbf{P}_A$ and $F_B \in \mathbf{P}_B$ be some other left transforming matrices of A and B. Then there exist matrices $H_A \in \mathbf{G}_{\mathrm{E}}$ and $H_B \in \mathbf{G}_{\Delta}$ such that $F_A = H_A P_A$, $F_B = H_B P_B$. Consider the following product of the matrices:

$$F_B F_A^{-1} = H_B P_B (H_A P_A)^{-1} = H_B P_B P_A^{-1} H_A^{-1} = H_B S H_A^{-1},$$

where $S = P_B P_A^{-1}$. Denote $H_B S = \|k_{ij}\|_1^n$. In view of Corollary 6 from [8], H_B has the form

$$H_B = \begin{Vmatrix} h_{11} & \dots & h_{1,n-1} & h_{1n} \\ \dots & \dots & \dots & \dots \\ h_{n-1,1} & \dots & h_{n-1,n-1} & h_{n-1,n} \\ \delta h_{n1} & \dots & \delta h_{n,n-1} & h_{nn} \end{Vmatrix}.$$

Hence,

$$k_{ni} = \left\| \delta h_{n1} \quad \dots \quad \delta h_{n,n-1} \quad h_{nn} \right\| \left\| \begin{matrix} s_{1i} \\ \vdots \\ s_{n-1,i} \\ s_{ni} \end{matrix} \right\| =$$

$$= \delta(h_{n1}s_{1i} + \dots + h_{n,n-1}s_{n-1,i}) + h_{nn}s_{ni} = \delta\ell_i + h_{nn}s_{ni}, \quad i = 1, \dots, k.$$

Consider the following greatest common divisor:

$$\left((\varepsilon,\delta),\, k_{n1}, \dots, k_{nk}\right) = \left((\varepsilon,\delta), \delta\ell_1 + h_{nn}s_{n1}, \dots, \delta\ell_k + h_{nn}s_{nk}\right) = d.$$

Since $(\varepsilon,\delta) \mid \delta$,

$$d = \left((\varepsilon,\delta), h_{nn}s_{n1}, \dots, h_{nn}s_{nk}\right) = \left((\varepsilon,\delta), h_{nn}(s_{n1}, \dots, s_{nk})\right).$$

The invertibility of H_B implies that $\left(\delta, h_{nn}\right) = 1$. Therefore, $\left((\varepsilon,\delta), h_{nn}\right) = 1$ and

$$d = \left((\varepsilon,\delta), k_{n1}, \dots, k_{nk}\right) = \left((\varepsilon,\delta), s_{n1}, \dots, s_{nk}\right).$$

Denote $SH_A^{-1} = \left\| t_{ij} \right\|_1^n$. Since $H_A^{-1} \in \mathbf{G}_{\mathrm{E}}$, according to Corollary 6 of [8], the matrix H_A^{-1} has the form

$$H_A^{-1} = \left\| \begin{matrix} v_{11} & v_{12} & \dots & v_{1k} & v_{1,k+1} & \dots & v_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_{k1} & v_{k2} & \dots & v_{kk} & v_{k,k+1} & \dots & v_{kn} \\ \varepsilon v_{k+1,1} & \varepsilon v_{k+1,2} & \dots & \varepsilon v_{k+1,k} & v_{k+1,k+1} & \dots & v_{k+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varepsilon v_{n1} & \varepsilon v_{n2} & \dots & \varepsilon v_{nk} & \varepsilon v_{n,k+1} & \dots & v_{nn} \end{matrix} \right\|.$$

Hence,

$$t_{ni} = \left\| s_{n1} \quad s_{n2} \quad \dots \quad s_{nn} \right\| \left\| \begin{matrix} v_{1i} \\ \vdots \\ v_{ki} \\ \varepsilon v_{k+1,i} \\ \vdots \\ \varepsilon v_{ni} \end{matrix} \right\| =$$

$$= s_{n1}v_{1i} + \dots + s_{nk}v_{ki} + s_{n,k+1}\varepsilon v_{k+1,i} + \dots + s_{nn}\varepsilon v_{ni}, \quad i = 1, \dots, k.$$

Consider

$$\begin{aligned} \left((\varepsilon,\delta), t_{n1}, \dots, t_{nk}\right) &= \left((\varepsilon,\delta), s_{n1}v_{11} + \dots + s_{nk}v_{k1} + s_{n,k+1}\varepsilon v_{k+1,1} + \dots + \right. \\ &+ s_{nn}\varepsilon v_{n1}, \dots, s_{nk}v_{1k} + \dots + s_{nk}v_{kk} + s_{n,k+1}\varepsilon v_{k+1,k} + \dots + \\ &+ \left. s_{nn}\varepsilon v_{nk}\right) = \left((\varepsilon,\delta), s_{n1}v_{11} + \dots + s_{nk}v_{k1} + \right. \\ &+ \varepsilon(s_{n,k+1}v_{k+1,1} + \dots + s_{nn}v_{n1}), \dots, s_{nk}v_{1k} + \dots + \\ &+ \left. s_{nk}v_{kk} + \varepsilon(s_{n,k+1}v_{k+1,k} + \dots + s_{nn}v_{nk})\right). \end{aligned}$$

In view of $(\varepsilon,\delta) \mid \varepsilon$, we get

$$\left((\varepsilon,\delta), t_{n1}, \dots, t_{nk}\right) = \left((\varepsilon,\delta), s_{n1}v_{11} + \dots + s_{nk}v_{k1}, \dots, s_{nk}v_{1k} + \dots + s_{nk}v_{kk}\right).$$

Since d is the divisor of all summands,

$$\left((\varepsilon,\delta), s_{n1}, \dots, s_{nk}\right) \mid \left((\varepsilon,\delta), t_{n1}, \dots, t_{nk}\right).$$

On the other hand, $S = \left\| t_{ij} \right\|_1^3 H_A$. Similarly, we show that

$$\left((\varepsilon,\delta), t_{n1}, \dots, t_{nk}\right) \mid \left((\varepsilon,\delta), s_{n1}, \dots, s_{nk}\right).$$

Hence,

$$((\varepsilon,\delta), t_{n1}, \ldots, t_{nk}) \mid ((\varepsilon,\delta), s_{n1}, \ldots, s_{nk}).$$

Application of the associativity of $M_n(R)$ completes the proof. ♦

Lemma 2. *Let*

$$\mathrm{E} = \mathrm{diag}\,(\underbrace{1,\ldots,1}_{k}, \underbrace{\varepsilon,\ldots,\varepsilon}_{n-k}), \qquad k = 1,\ldots,n-1, \qquad \Delta = \mathrm{diag}(1,\ldots,1,\delta),$$

and $U = \|u_{ij}\|_1^n \in GL_n(R)$. *Then*

$$\mathrm{E}U\Delta \sim \mathrm{diag}\,(\underbrace{1,\ldots,1}_{k-1}, \underbrace{\mu_k,\ldots,\mu_n}_{n-k+1}), \qquad k = 1,\ldots,n-1.$$

P r o o f. Consider the product of matrices

$$\mathrm{E}U\Delta = \left\| \begin{array}{ccc|c} u_{11} & \ldots & u_{1,n-1} & \delta u_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{k1} & \ldots & u_{k,n-1} & \delta u_{kn} \\ \hline \varepsilon u_{k+1,1} & \ldots & \varepsilon u_{k+1,n-1} & \varepsilon\delta u_{k+1,n} \\ \ldots & \ldots & \ldots & \ldots \\ \varepsilon u_{n1} & \ldots & \varepsilon u_{n,n-1} & \varepsilon\delta u_{nn} \end{array} \right\| = \left\| \begin{array}{cc} U & V \\ U' & V' \end{array} \right\|.$$

Since the matrix U has the size $k \times (n-1)$, where $k + n - 1 \geq n$, $k = 1,\ldots,n-1$, and it is a submatrix of the invertible matrix of order n, by Proposition 3.7 from [3]

$$U \sim \mathrm{diag}(\underbrace{1,\ldots,1}_{k-1}, u),$$

where $u \in R$ is not a divisor of unit and $k = 1,\ldots,n-1$.

Lemma is proved. ♦

Theorem 1. *Let R be a commutative principal ideal domain and let*

$$A \sim \mathrm{diag}\,(\underbrace{1,\ldots,1}_{k}, \underbrace{\varepsilon,\ldots,\varepsilon}_{n-k}), \qquad k = 1,\ldots,n-1, \qquad B \sim \mathrm{diag}\,(1,\ldots,1,\delta).$$

Then the Smith normal form of the least common right multiple of matrices A and B has the form:

$$\Omega = \mathrm{diag}\left(\underbrace{1,\ldots,1}_{k-1}, \frac{(\varepsilon,\delta)}{((\varepsilon,\delta), s_{n1},\ldots,s_{nk})}, \underbrace{\varepsilon,\ldots,\varepsilon}_{n-k-1}, [\varepsilon,\delta] \right).$$

P r o o f. Remark that according to Lemma 1, the element $((\varepsilon,\delta), s_{n1},\ldots,s_{nk})$, and, hence, the matrix Ω do not depend on the choice of transforming matrices P_A and P_B.

By Theorem 2 of [1], the Smith normal form of the greatest common left divisor of the matrices A and B has the form

$$(A,B)_\ell \sim \mathrm{diag}\,(1,\ldots,1,(\varepsilon,\delta,s_{n1},\ldots,s_{nk})).$$

According to Theorem 1.20 from [3, p. 53], the least common right multiple of matrices A and B can be written as $[A,B]_r = BUA_1$, where $A = (A,B)_\ell A_1$, $U \in GL_n(R)$. Since the matrices B and A_1 have the Smith normal forms $\mathrm{diag}\,(1,\ldots,1,\delta)$ and $\mathrm{diag}\,(\underbrace{1,\ldots,1}_{k}, \underbrace{\varepsilon,\ldots,\varepsilon}_{n-k})$, respectively, by Lemma 2,

$$[A,B]_r \sim \mathrm{diag}\,(\underbrace{1,\ldots,1}_{k-1}, \underbrace{\omega_k,\ldots,\omega_n}_{n-k+1}).$$

In view of Theorem 1.18 from [3, p. 49], we obtain

$$\pm\det A \det B = \det\,(A,B)_\ell \det\,[A,B]_r,$$

i.e.

$$\det[A,B]_r = \pm\frac{\det A \det B}{\det(A,B)_\ell} = \pm\frac{\varepsilon^{n-1}\delta}{(\varepsilon,\delta,s_{n1},\dots,s_{nk})} = \omega_k\omega_{k+1}\dots\omega_{n-1}\omega_n .$$

It follows from [9] that $\omega_n = [\varepsilon,\delta]$ and $\omega_i \mid \varepsilon$, $i = k+1,\dots,n-1$. Since $\mathrm{E} \mid \Omega$, we get $\varepsilon \mid \omega_i$, $i = k+1,\dots,n-1$, which implies $\omega_i = \varepsilon$, $i = k+1,\dots,n-1$. Hence,

$$\omega_k = \pm\frac{\varepsilon^{n-k}\delta(\varepsilon,\delta)}{\varepsilon^{n-k-1}\varepsilon\delta(\varepsilon,\delta,s_{n1},\dots,s_{nk})} = \pm\frac{(\varepsilon,\delta)}{(\varepsilon,\delta,s_{n1},\dots,s_{nk})}.$$

Taking into account that the invariant factors of matrix are chosen accurate to within divisors of unit, we obtain that the Smith normal form of the least common right multiple of matrices A and B has the form

$$\Omega = \mathrm{diag}\left(\underbrace{1,\dots,1}_{k-1},\frac{(\varepsilon,\delta)}{((\varepsilon,\delta),s_{n1},\dots,s_{nk})},\underbrace{\varepsilon,\dots,\varepsilon}_{n-k-1},[\varepsilon,\delta]\right).$$

Theorem is proved. ♦

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НОРМАЛЬНА ФОРМА СМІТА НАЙМЕНШОГО СПІЛЬНОГО КРАТНОГО МАТРИЦЬ ДЕЯКОГО КЛАСУ МАТРИЦЬ

Для неособливих матриць довільного порядку над областю комутативних головних ідеалів, при деяких обмеженнях на їхні канонічні діагональні форми, побудовано нормальну форму Сміта найменшого їхнього спільного правого кратного.

Ключові слова: *область головних ідеалів, нормальна форма Сміта, найменше спільне праве кратне матриць.*

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